CDO Pricing: Copula Implied by Risk Neutral Dynamics

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Abstract

When dealing with multi-issuer credit derivatives such as CDO, it is customary to refer the reader to either of two approaches: "static models" which focus on the copula between the variables of interest, and "dynamic models" where the diffusion of the underlying variables is described directly. While the former is widely used due to its simplicity, it is not clear that there is a well behaved dynamic model consistent with a given static approach. For this reason, it is often argued that an understanding of the dynamics used in model for CDO is required to bring it to par with derivative models used for other asset classes, such as the risk neutral diffusion models used for equity, currency and commodity options derived from Black and Scholes, or the characterization of arbitrage free term structure of interest rates obtained by HJM.

Clearly, a "dynamic model" implies a certain copula between the random variables of interest. The goal of this article is to develop a unified view compatible with both approaches, and reach a better understanding of the properties that a good "dynamic model" used for pricing and hedging would have when seen as a static model.

We focus on credit models where large homogeneous pool portfolio are mathematically possible, a common assumption among practitioners. In a general credit term structure dynamics framework similar to HJM, we identify a "systemic loss" process linked to the survival dynamics that allows to identify the density of loss for large portfolios, and to explicit the default copula between the issuers. We then apply these results to different classes of CDO models that have been put forward for their tractability, to see what copula is implied by given a dynamic model, and what dynamic models could give rise to some popular copula model. The three classes we review are the one factor copula models, the markovian loss intensity models, and the systemic intensity jump diffusion models.

Keywords: Credit Model, Default Intensity, HJM, CDOs, Portfolio Effects, Factor Copula
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1 Introduction and Motivation

1.1 The need for a unified view on CDO pricing

The static and the dynamic approaches come from different interpretative framework: copula have been used by insurance actuaries to model the probability of joint events using statistics to estimate probabilities. This contrasts with most derivatives pricing literature, where martingale theory, and typically a brownian motion is used to model of the evolution of market variables to setup a hedging strategy.

These approaches are usually exposed and discussed separately. Both have so far proven such fertile areas of research that many results can be stated by focusing exclusively on only one of them. Their demarcation has become so entrenched that for a casual observer, these two separate streams of research are vying for the same goal through antagonist means.

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The portfolio loss model of KMV [3] and Credit Metrics [4] are typical examples of the static models[1]. They were developed to assess portfolio effects in credit for financial institutions based on historical losses or rating migrations long before the credit market developed and gave the possibility to hedge some of that risk. A typical representant of the second category would be correlated intensity diffusion models such as affine jump diffusion[2] proposed by Duffie and Garleanu [7].

Yet, a "dynamic model" implies a certain copula between the random variables of interest, which means that a link exists between the two approaches. As the prevalent market practice is to use a static copula approach with market implied probability for pricing and hedging, the link between the two approaches needs to be explicited to better understand P/L on a hedged trade. This is what Fermanian and Vigneron [13] set themselves up to do for the gaussian copula. The link is also needed to improve the understanding of the high level properties of dynamic model: translating them into pricing copula helps to see whether the properties make sense or not..

As we feel there is already much to say about unifying these two views on default dependence, we will not deal with recovery modeling or actual CDO payoff here, but deal only with the stylised case of obtaining the distribution of the portfolio loss at a given time horizon.

1.2 A critical look at copula modeling in credit

1.2.1 Early Phase

Early models of the portfolio loss distribution, such as KMV [3] used a dynamic model of the asset value to arrive at a portfolio loss distribution at a given time horizon. The interpretation in terms of assets of these models clearly inherit from "structural" models such as Merton [1]. The latter relied on accounting figures and stock price, but nothing prevents a more suitable source of default probabilities estimates to be used to calibrate the "distance to default" relevant to the model. This was in fact done for CDO as soon as market implied default probabilities were available.

The model used in these cases is essentially a single time horizon model. As CDOs can be expressed as a linear combination of single time horizon options on the loss, they can be priced by using a set of separate single time horizon models. This point makes the presentation of this pricing approach more arduous, as we leave the sanitised world of describing one model, and have to (a) show that the payoff of a CDO is indeed a linear combination of options on loss at different times and (b) explain when it is acceptable to price two separate options with related underlyings with two completely separate models.

1.2.2 The age of default time copula

In the seminal article introducing the concept of copula to the field of CDO pricing, D. Li [6] departs from the modeling of losses at a single time horizon and introduces the copula of default time. This results in a loss model that is intrinsically consistent for all time horizons. The previous approach, which involves both due consideration of the CDO payoff and some meta-modeling discussion on the adequacy of using different option models for two unrelated underlying was apparently no longer needed.

The idea of adding dynamics to copula of default time was investigated in its most general form by Schönbucher and Schubert [8]. This was a major contribution to the understanding of how default time copula impact dynamics, in particular, of how a choice of copula determines intensity jumps. It will be convenient for our purpose to quickly recall the approach and results obtained there: the model defines "pseudo-intensities" (default intensity given a filtration $\mathcal{F}_t$ that does not contain the default times) and introduces defaults dependence through a copula of default times conditional on the trajectory of these pseudo-intensities. In this model, defaults

[1] Both result in the same loss distribution as the gaussian copula. The CreditRisk+ model, developed around the same time as Credit metrics corresponds to the Clayton copula.

[2] As instantaneous correlation of the intensity diffusion was unable to reach the required levels of terminal correlation when using normal to lognormal credit diffusions, many have attempted to add common jumps to their model to boost the terminal correlation

bring information on the probabilities of other issuers default, and cause intensities observed in the market (intensities given both $\mathcal{F}_t$ and all past default information) to jump: the $\mathcal{G}_t$ default intensities (those that are observable in the market filtration) combine information on pseudo-intensity trajectory and default realisation. The resulting "unified model" can produce a specified default time copula as a special case when pseudo-intensities are determinist, or an arbitrary intensity diffusion model when the conditional copula is the independence copula. As noted by Roncalli [9], the model is much more general than any dynamic model proposed until then, as all previous dynamic models assumed independent default conditional on the trajectory of intensities.

1.2.3 Deconstruction of default time copula models

These default time copula with dynamics models are more general than either approach. The generalisation leads to the emergence of undesirable properties that are not present in the two approaches it attempts to unify. First, intensity jumps of unlimited size can be obtained from a mundane gaussian copula. Second, in the "unified" model proposed above, an issuer can not be added to the credit universe without altering the dependence structure of existing issuers, and hence the price of a CDO, this was not the case in either static or dynamic models, This led to conclusion that something was indeed wrong with the dynamic properties of default time copula models.

When a model is beset by difficulties, there is value in reconsidering the assumptions made. As it turns out, these undesirable properties come from the intertemporal dependence of defaults implied by the copula of default times assumption. The assumption that a copula of default times can be chosen a priori invites the proposition that the default time copula does not change over time, but the latter assumption turns out to have fundamental consequences on the information structure: the model accumulates information about the future as time goes by, and is eminently non stationary. It also determines the size of intensity jumps on default (and as it turns out, the jump size gets smaller as the number of issuers in the credit universe increases). This contrasts with earlier one period models, where the jump size is not determined by the copula of default indicator used. As discussed in Y. Li [12], only the copula of default indicators needs to be known to price CDO, while the prescription of the copula of default times, for all the apparent simplicity it brings, has further implications on intertemporal dependence of defaults. This distinction is not important when considering the model statically, but it becomes crucial when one considers such questions as martingality.

1.3 The way for a unified view on CDO pricing

We therefore propose that copulas should be used in this field as a phenomenological device, to "describe" a loss distribution, rather than as an explanatory tool into the cause of dependence. Such a unified view is best achieved by specifying dependence through realistic default probability dynamics, and observing what default indicator copula is implied by these dynamics, rather than using the copula of default time as a causal driver of dependence.

We set out to work here on a unified view, where the default dependence is caused solely by the risk neutral dynamics of default probabilities in the market filtration, and the copula is implied by these dynamics. Instead of specifying pseudo-intensities, we directly model intensities in the market filtration so that the evolution of survival probability and hazard rates can be described in the market filtration directly as in HJM for rates. Instead of specifying a default time copula conditional on the pseudo-intensities trajectories, we consider the resulting copula of default indicator, which is the one used in the the static approach.

1.4 Outline of this document

In section 2, we first give a definition of the "background intensity" used throughout, a natural prolongation of the default intensity process which we will need to define hazard rates in the market filtration. Unlike arbitrary $\mathcal{F}_t$ intensities based definitions, this definition is well anchored in the market filtration and corresponds to the default intensity before default. Following that, we derive the General Credit Term Structure Dynamics (2.8) in the market filtration. Those equations are closely linked to the usual credit HJM formulae. While the HJM model is usually aimed at showing the arbitrage free drift condition on hazard rates (resp. forward rate for interest rate HJM), our presentation puts emphasis on the martingality of the survival probability diffusion in the market filtration, taking past defaults into account, which is essential for portfolio loss modeling.

In section 3, we give a link between our stated requirement that arbitrarily many exchangeable issuers can be added to our setup and our no contagion assumption. We take advantage of the definition of canonical background information in section (2) to give these terms an unambiguous meaning as well.

In section 4, we establish diversification properties by identifying the systemic loss process (4.9). To build intuition, this is first done by studying intensity diffusion models where a systemic and idiosyncratic intensity evolve independently, extending them to general intensity diffusion and jump diffusion models. We explicit the dynamics of systemic loss process, which is the diffusing variable driving the portfolio loss. These results allow to understand what density is obtained at t for spot and forward losses to time T, and how it diffuses. We then look at the constraints imposed on this variable dynamics by CDO market data.

In section 5, we study whether the variables of interest are independent conditional on the systemic loss process. We then obtain what is effectively an extension of the De Finetti's theorem when issuers are not exchangeable. We explicit the random variables conditional on which defaults are independent, which give us a natural copula factor implied by the General Credit Term Structure Dynamics (integrated form) (2.10), and allow us to specify the Canonical Copula. We show that although there is an infinity of copulas that can accomodate a given joint distribution, in the general case, the default indicator copula will tend towards the n factor copula implied by the dynamics. The pricing copula can be reduced to one factor only in specific cases like homogeneous portfolio or degenerate dynamics. While this is a negative result, this allows us to better understand the fundamental properties of the CDO pricing models proposed so far, which by design correspond to 1 factor copula pricing problems.

In section 6, we review some of the most popular CDO models, and the way their implied factor structure influences their properties. The three classes we review are the one factor copula setup, the intensity jump diffusion models, and the markovian loss intensity models.

2 Model Setup

2.1 Section overview

This section first gives a definition of the "background intensity", a natural prolongation of the default intensity process which we will need to define hazard rates in the market filtration. It states Hypothesis (HH1), which seems quite reasonable and for which we have no counterexample yet, and then derives the General Credit Term Structure Dynamics (2.8) in the market filtration. Those equations clearly explicit the survival probability term structure in the market filtration, properly taking past defaults into account.

Then, as we think a good framework should provide results that hold when arbitrary large sequence of exchangeable names can be added, we state Hypothesis (H2).

The results in the next sections are valid for any model for which the General Credit Term Structure Dynamics (2.8) holds in the market filtration. As many avenues of research are being actively pursued at the moment, we would not want this result to be limited to stopping times that are the first arrival time of a poisson process. Instead, we seek the most general definition of hazard rate for a stopping time, one that covers reduced form and structural models, as well as some forms of contagion.

2.2 Canonical definition of single name background information

2.2.1 Background information definition strategy

In this section, we recall that the default intensity, which we will note $\tilde{\lambda}_t^i$, goes to zero after the default. To write hazard rates and intensity dynamic equations that are valid irrespective of the default time realisation, we need an extension of the default intensity that does not go to 0 after the default, just as would be the case if τ^i was the first arrival time of a poisson process. This extension makes general default time model formalism correspond to the "reduced form model". We call this extension the "background intensity" λ_t^i.

The concept of background intensity is already well known in credit, but up to now, there seemed to be as many of them as arbitrary subfiltration $\mathcal{F}_t \subset \mathcal{G}_t$ choices. The proposed setup complements results stated in existing credit research where typically, a subfiltration $\mathcal{F}_t \subset \mathcal{G}_t$ is "chosen" arbitrarily, and a set of technical properties that seem no less arbitrary is listed. What we explicit here a further condition to ensure that the survival dynamics implied by the background intensity coincide with the survival dynamics in the market filtration $\mathcal{G}_t$ for $\tau^i > t$.

Our condition is basically that *the background filtration is independent from default realisation markers* $\tilde{\theta}^i$. This hypothesis accommodates reduced form and structural models, and can deal with contagion. On top of that, its intuitive comprehension and interpretation is easier than with usual technical conditions.

We will introduce definitions in that order:

$$\mathcal{G}_t \rightarrow \tau^i \rightarrow \tilde{\lambda}_t^i \rightarrow \tilde{\theta}^i \rightarrow \mathcal{G}_t^{-i} \rightarrow \lambda_t^i$$

In a sense, we orthogonalise the information embedded in $(\tau^i, \tilde{\lambda}_t^i)$, the default time and default intensity, into $(\tilde{\theta}^i, \lambda_t^i)$, which are respectively the default realisation marker variable (the transform of the default time into a uniform variable indicating given the intensity trajectory), and the background intensity. The latter two are independent by construction. Then, we can study the correlations between the λ_t^i, which in turn will imply correlations between the defaults.

We will after that be in a position to derive the General Credit Term Structure Dynamics in the market filtration.

Let's now describe the model setup in technical terms.

2.2.2 Single name background information definition

We consider a filtered probability space $(\Omega, \mathcal{G}_t, P)$ where the filtration $\mathcal{G}_t$ can be generated from a finite number of brownian and poisson process drivers, and all the conditions for the martingale representation theorem are assumed to be met. Thus an event $\omega \in \Omega$ consists of the trajectories of the brownians and the poisson process. This is a market filtration in so far that all the asset prices under consideration and the usual numeraires are $\mathcal{G}_t$ adapted, and the default times τ^i of any issuer i are $\mathcal{G}_t$ stopping times. We introduce the default indicators $\tilde{N}_t^i := 1_{\{\tau^i < t\}}$ and the default intensities[3] $\tilde{\lambda}_t^i$:

$$\tilde{\lambda}_t^i dt := E(d\tilde{N}_t^i | \mathcal{G}_t) \tag{2.1}$$

As $\tilde{N}_t^i := 1_{\{\tau^i < t\}}$, unlike a poisson process, can only jump once, the intensity $\tilde{\lambda}_t^i$ jumps to 0 upon default. This means that $\tilde{\lambda}_t^i$ becomes a very different kind of random process once it hits the default time τ_i: one that stays at 0.

In order to define unconditional hazard rates, we need a natural extension of intensity after default, one that does not depend on default realisation. To formalise the notion of removing default realisation information, we introduce the default realisation marker $\tilde{\theta}^i$, which is the transform of the default time variable into a uniform $U(0,1)$ variable conditional on the path of

[3] We use here a differential notation to keep definitions similar to those of poisson processes, this still corresponds to an integral definition with default indicator and compensator. The default intensity is therefore defined as the pathwise derivative of the default compensator with respect to t in the distribution sense. As the compensator is a pathwise monotonic function, it is differentiable almost everywhere, and $\tilde{\lambda}_t^i$ has at most a countable set of positive diracs.

default intensity. We detail its general definition in appendix A.1.1, for illustrative purpose, in a "usual" intensity model, we have:

$$\tilde{\theta}^i = \exp\left(-\int_0^{\tau^i} \tilde{\lambda}_t^i dt\right) \tag{2.2}$$

If the default time is the first jump of a poisson process, this variable is independent of the poisson intensity. It contains the information needed to obtain τ^i when knowing the poisson intensities $\lambda_s^i, s < t$.

As we want to define a background intensity that is independent of the default realisation, we now focus on the subfiltrations $\mathcal{G}_t^{-i}$ of $\mathcal{G}_t$ independent of $\tilde{\theta}^i$. Their existence and properties are detailed in appendix A.1. We can then define the background intensity λ_t^i, by projection onto the background filtration[4]:

$$\lambda_t^i := E(\tilde{\lambda}_t^i | \mathcal{G}_t^{-i}, \tau^i > t) \tag{2.3}$$

We now have the required setup to state the following assumption concerning the information available in the $\mathcal{G}_t^{-i}$ we selected on the stopping time τ^i:

> **Hypothesis (HH1): Default Times Background Intensity**
> *There exists a filtration $\mathcal{G}_t^{-i}$ independent of $\tilde{\theta}^i$ such that the background intensity $\lambda_t^i := E(\tilde{\lambda}_t^i | \mathcal{G}_t^{-i}, \tau^i > t)$ coincides with $\tilde{\lambda}_t^i$ on $\{\tau^i > t\}$.*

Intuitively, we see that this background intensity λ_t^i effectively carries information after default[5], whereas $\tilde{\lambda}_t^i$ jumps to 0 at the default time, and therefore no longer "carries" any background information.

There is more to say on the technical properties of this canonical setup, but these properties can be summarised as being the same usual ones encountered with reduced form models. We refer the reader to the appendix section A.1 for further discussion and proof of the properties obtained in this setup.

2.3 General credit dynamics for single name

We can now proceed with the derivation of the General Credit Term Structure Dynamics (2.8) equation, which is the extension of the HJM equation to the market filtration $\mathcal{G}_t$ thanks to (HH1). The emphasis here is on correctly capturing the default realisation, something that HJM calculation usually conditional on $\tau^i > t$, or on a filtration without default could not do. This "extension" ensures that past defaults are properly taken into account, and results in survival probabilities being martingales instead of drifting by λ_t^i. Our goal is to study the evolution of the survival probability as observed in the market filtration $\mathcal{G}_t$:

> **Definition: Survival Probability and Hazard Rates**
>
> $$P_t^i(T) := E\left(1_{\{\tau^i > T\}} \middle| \mathcal{G}_t\right) \tag{2.4}$$
>
> $$\exp\left(-\int_t^T h^i(t,u)du\right) := E\left(\exp\left(-\int_t^T \lambda_u^i du\right) \middle| \mathcal{G}_t\right) \tag{2.5}$$

The definition (2.5) actually extends the definition of hazard rates after default. As we show in appendix A.1 that:

$$P_t^i(T) = 1_{\{\tau^i > t\}} E\left(\exp\left(-\int_t^T \lambda_u^i du\right) \middle| \mathcal{G}_t\right) \tag{2.6}$$

[4]This is the usual way of defining a $\mathcal{F}_t$ intensity given any subfiltration $\mathcal{F}_t \subset \mathcal{G}_t$, what is specific here is that we explicit the canonical choice of background filtration.

[5]Until such time t as $\int_0^t \lambda_s^i ds = +\infty$, which means that default is almost sure.

It corresponds to the usual definition of hazard rates if hypothesis (HH1) is verified.

It is possible to derive the equation (2.8) without hypothesis (HH1) by using an arbitrary background filtration $\mathcal{F}_t$. Ie this case, the hazard rates describe the evolution of $E(P_t^i(T)|\mathcal{F}_t)$ and there seem to be as many hazard rates as background filtration choices. This approach has the benefit of giving a definition to hazard rates in the market filtration, which is what can be observed.

We write the evolution of hazard rates[6]:

$$dh^i(t,T) = a^i(t,T)\,dt + \sigma^i(t,T)\,.dW_t^i$$

This is the transposition of HJM notations to credit, as $\mathcal{G}_t$ is generated by brownian and poisson processes, the evolution $h^i(t,T)$ can in all generality be decomposed in terms of drift, brownians and poisson increments. We first present here hazard rates that follow a general brownian diffusion similar to HJM, mainly because it is a notationally simpler setup, where the analogy with rates works well. However, this assumption is not essential[7], and will in fact be relaxed in section 4.4, where we consider the most general semi-martingale representation of hazard rates by adding jumps[8].

Looking back at the $\mathcal{G}_t$ local martingale decomposition of $P_t^i(T)$, we have:

$$\begin{aligned}
P_t^i(T) &= 1_{\{\tau^i>t\}} \exp\left(-\int_t^T h^i(t,u)du\right)\\
\frac{dP_t^i(T)}{P_{t^-}^i(T)} &= \frac{1}{2}\left|v^i(t,T)\right|_\rho^2 dt - A^i(t,T)\,dt - v^i(t,T)\,.dW_t^i + \tilde{\lambda}_t^i dt - d\tilde{N}_t^i\\
v^i(t,T) &:= \int_t^T \sigma^i(t,u)du\\
A^i(t,T) &:= \int_t^T a^i(t,u)du
\end{aligned}$$

The definition (2.4) implies that $P_t^i(T)$ is a $\mathcal{G}_t$ martingale[9], leading to a condition on the hazard rates drift:

$$a^i(t,T) = v^i(t,T)\,.\rho\sigma^i(t,T)$$

We can now state the diffusion equation for hazard rates and survival probabilities:

General Credit Term Structure Dynamics:

$$dh^i(t,T) = v^i(t,T)\,.\rho\sigma^i(t,T)\,dt + \sigma^i(t,T)\,.dW_t^i \tag{2.7}$$

$$\frac{dP_t^i(T)}{P_{t^-}^i(T)} = -v^i(t,T)\,.dW_t^i + \tilde{\lambda}_t^i dt - d\tilde{N}_t^i \tag{2.8}$$

[6] As in a usual general HJM framework, we will use the following notations: W_t^i is a $\mathcal{G}_t$ adapted multidimensional brownian, $\sigma^i(t,T)$ is a $\mathcal{G}_t$ adapted stochastic volatility vector with the same dimension as the brownian dW_t^i. We denote here by ρ the $\mathcal{G}_t$ adapted instantaneous correlation matrix of dW_t^i, and $|X|_\rho^2 = \sum_{i,j} X_i \rho_{ij} X_j$.

[7] We introduce assumptions as we need them. Regularity conditions on the intensity are often cited in the credit litterature, but such assumptions seem at first overcautious, since measure theory grants us the existence of intensity pathwise for almost every t. (Again, the existence of intensity is granted from measure theory: as the derivative in the sense of distributions of a pathwise monotonic function, the compensator exists almost everywhere.) In such instance, it is instructive to enquire about what happens when intensity regularity assumptions are not met, because genuine credit model correspond to such case. For instance, if the compensator is driven by an infinite activity Levy processes, such a variance gamma, the pathwise intensity $\tilde{\lambda}_t(\omega)$ is null almost t everywhere, but, what happens on this null measure set actually matters a lot. Still, we take note that Levy compensators can be obtained as a limit of poisson jumps, so that the study of jump diffusion behavior can be taken as the start for a more general study, in later work. Bearing that in mind, the relaxation of this assumption would not bring something essentially different.

[8] We will see that introducing jumps does indeed introduce many new notations and questions, some of the latter concerning completeness and robustness of the model we can not answer, but that the concepts introduced here to analyse dependence introduced here still apply

[9] Since $E\left(1_{\{\tau^i>T\}}\middle|\mathcal{G}_t\right) = E\left(E\left(1_{\{\tau^i>T\}}\middle|\mathcal{G}_u\right)\middle|\mathcal{G}_t\right)$ we have $P_t^i(T) = E\left(P_u^i(T)\middle|\mathcal{G}_t\right), \forall u \geq t$

The first equation is the same as the HJM formulae for credit, our extension of hazard rates after default did not impact their evolution equation. Credit literature often shows the diffusion of $E(P_t^i(T)|\tau^i > t)$ where the poisson term is absent, and there is a drift of λ_t^i. The $\mathcal{G}_t$ version, where $P_t^i(T)$ is a martingale, will be useful to us as it explicitly takes into account defaults occurring between 0 and t. The latter equation, can be written in integral form to show the various martingale terms:

$$P_t^i(T) = P_0^i(T) \exp\left(-\int_0^t \frac{1}{2}\left|v^i(s,T)\right|_\rho^2 ds + v^i(s,T).dW_s^i\right) \exp\left(\int_0^t \tilde{\lambda}_s^i ds\right) 1_{\{\tau^i > t\}} \tag{2.9}$$

We use the Doléans exponential to simplify notations, it can be defined for any local martingale X by:

$$\varepsilon(X) := \frac{\exp(X)}{E(\exp(X))}$$

General Credit Term Structure Dynamics (integral form):

$$P_t^i(T) = P_0^i(T)\, \varepsilon\left(-\int_0^t v^i(s,T).dW_s^i\right) \exp\left(\int_0^t \tilde{\lambda}_s^i ds\right) 1_{\{\tau^i > t\}} \tag{2.10}$$

We refer the advanced reader to consult the appendix section where measure changes arguments are used to get further insight into the properties of this model:

Section A.3 summarises the key assumption, notations and properties for risk neutral pricing. While this whole paper can be read naively under the natural probability measure, our goal is to describe CDO pricing by replication. Therefore, we had to explicit the conditions for existence and unicity of risk neutral pricing measures for credit.

We then propose a deeper look at the comparison of general credit dynamics stated here and conditional diffusion published previously as "Credit HJM" in section A.4, we also expound a bit on the comparison to the rates HJM setup.

Finally, section A.5 recalls some results linking hazard rates and intensities, so that we can get some more intuition on the hypothesis (HH1), by remarking that it is linked to the question whether the hazard rate volatility $\sigma^i(t,T)$ is independent of the default realisation embedded in $\tilde{\theta}^i$.

While much more can be said on this single name setup, we decided—for our purpose which is to study multi-name effects—that it was best to give here the bare minimum introduction in terms of setup and notation, with a few hints at what motivates our choices.

2.4 Conclusion on single name $\mathcal{G}_t$ dynamics

In this section, we have presented a definition of the default intensity, which is *a process eminently dependent on filtration and probability.* We have then formalised with (HH1) the concept of background intensity which allowed us to describe the dynamics of the $\mathcal{G}_t$ survival probabilities, which yields essentially the same result on hazard rate drift as credit HJM, but takes default into account for the survival probability evolution.

Those results were already available for defaults times constructed as first jump of poisson process whose a intensity was defined in a background filtration $\mathcal{F}_t$. Our hypothesis (HH1) allows to show that they are also available for a much wider class of stopping time. We presented this here to make the paper more general[10] and more self contained, while having a *rigorous definition of what constitutes "background" information*: something that will help when considering multi-name setup.

In the next section, we will leave the domain of single issuer probability modeling and make full use of the no contagion hypothesis, to which we also give an explicit technical definition, to study diversification effect in this setup.

[10]For instance, we will see that the results given here apply not only for totally unexpected default models, but also for the KMV model described in 6.1.2 for which the default indicator have a gaussian copula

3 Multi-Name Setup and No Contagion Assumption

3.1 Disambiguating conditionally independent defaults and no contagion

Before looking into the dynamics of portfolio loss, we need to explicit the important assumption that results in conditional independence. As a consequence, the setup can be summarized as one where *hazard rates evolve independently from the default time realisations.* The following hypothesis used in this paper imply something fundamental about the default dependence structure when there are multiple issuers: hazard rates do not jump upon default. Since contagion[11] specifically involves a jump in survival probabilities when other issuers default, there is *no contagion.*

We choose this because we want our credit models to be able to *handle arbitrary large pool of exchangeable issuers*, because *we do not want it to change* if we have to increase the portfolio size. Indeed, if we were using contagion models where issuers are exchangeable, the hazard rate jumps on default would tend to 0 as the number of credit issuer is increased, the absence of default contagion is anyway the large pool limit for exchangeable issuers.

Our hypothesis (HH2) states that the $\tilde{\theta}^i$ are independent, and the λ_t^i are $\mathcal{F}_t$ adapted. That independence hypothesis, which seems plainly acceptable when modeling financial instruments and allows enough dependency structures, causes diversification in large portfolios and avoids the undesirable properties identified by Roncalli [9].

In a multi-name setup, we define the background filtration $\mathcal{F}_t$ as a maximal subfiltration of $\mathcal{G}_t$ independent of all issuers default realisations $(\tilde{\theta}^1, ...\tilde{\theta}^n)$:

To preserve our ability to handle arbitrarily large pool of exchangeable issuers, we now state our technical hypothesis for no contagion:

> **Hypothesis (HH2): No Contagion**
> *The default realisation markers* $\tilde{\theta}^i := \exp\left(-\int_0^{\tau} \tilde{\lambda}_s^i ds\right)$ *are independent, and there exists a filtration* $\mathcal{F}_t$ *independent of all* $\tilde{\theta}^i$ *such that the background intensity* $\lambda_t^i := E(\tilde{\lambda}_t^i | \mathcal{F}_t, \tau^i > t)$ *coincides with* $\tilde{\lambda}_t^i$ *on* $\left\{\tau^i > t\right\}$.

Assumption (HH2) implies something fundamental about the default dependence structure when there are multiple issuers: hazard rates do not jump on other's default time.

It should be noted that (HH1) alone is compatible with contagion: the intensities λ_t^i are adapted to the subfiltration $\mathcal{G}_t^{-i}$, but may still jump on default of other issuers $j \neq i$. In a contagion model, the filtration $\mathcal{F}_t$ is called the pseudo-intensity filtration, and the $\mathcal{G}_t^{-i}$ adapted λ_t^i which jump on other issuers' default are *not* adapted to $\mathcal{F}_t$.

Another remark concerning the term "no contagion" and "conditionally independent defaults": we see here two possible causes of contagion. One which is the most fundamental, and the most toxic[12] corresponds to the $\tilde{\theta}^i$ being dependent. The "conditionally independent defaults" setup corresponds to the case where the $\tilde{\theta}^i$ are independent, and avoids the undesirable properties linked with the introduction of a copula between the $\tilde{\theta}^i$. However, such a setup could still have contagion through intensities jumps on default. If, in addition to that, one requires that all the background intensities be measurable in a common background filtration $\mathcal{F}_t$, intensities can not jump on default, and we get the assumption (HH2), which is what is needed to have "no contagion". In the following we refer to the latter as a no contagion hypothesis, to avoid confusion with a model where default realisations $\tilde{\theta}^i$ conditional on the λ_s^i—which are not necessarily adapted to $\mathcal{F}_t$—are independent.

[11] A technical definition of what we mean by contagion follows in this section

[12] The information of how early a very risky name defaults gives information on the probability of default of a very low risk name likelihood of default in, say two thousand years than in one thousand years.

Property: Conditional Independence and No Contagion under (HH2)
The default indicators $1_{\{\tau^i<t\}}$ are independent conditional on $\mathcal{F}_t$. Hazard rates do not jump on default, so that default correlation comes solely from the joint movements of $\mathcal{F}_t$ adapted intensities.

The formal proof of this property is given in appendix section A.2. Again, it coincides with the constructive approach.

We argued earlier that contagion—hazard rate jumps on defaults—is not consistent with the assumption that arbitrarily many exchaneable names can be added in the model setup, this leads us to the following conjecture, for which we do not have a full proof yet :

Conjecture: (HH1) and (HH2) hold if names are part of arbitrarily large exchangeable sequences
If for each issuer i, a sequence of arbitrarily large p_i exchangeable issuers can be found, then (HH2) holds.

3.2 Conclusion on no contagion hypothesis

In an attempt to create a framework in which *arbitrary large sequence of exchangeable names can be added*, we had to work on a precise definition of no contagion.

We thus stated Hypothesis (HH2) to obtain that property, and even think that the possibility of adding arbitrary large sequence of exchangeable names can be added is enough to have (HH1) and (HH2) hold.

4 Diversification Effects on the Loss of a Portfolio

In this section, we show the link between survival probability dynamics and diversification properties. These calculations are made possible by our previous specifications, ie HJM in the market filtration and (HH2) hypothesis. We start with a simpler case of independent systemic and idiosyncratic diffusion to build intuition.

Notation: Systemic Random Drivers and Information
We decompose the brownian vector dW_t^i onto an independent base $\left(dW_t^{ck}, dW_t^{\epsilon il}\right)$ with $k \in \{1,...p\}, l \in \{1,...q\}, i \in \{1,...n\}$ such that the $\left(dW_t^{\epsilon il}\right)_{l\in\{1,...p\}}$ are "idiosyncratic" factors independent from the evolution of survival of other issuer $j \neq i$. The remaining $\left(dW_t^{ck}\right)_{k\in\{1,...p\}}$ are "common"factors that participate to the evolution of several issuers:

$$dW_t^i = \sum_{k=1}^{p} \beta_k^i dW_t^{ck} + \sum_{k=1}^{q} \beta_{p+k}^i dW_t^{\epsilon ik}, \quad \sum_{k=1}^{p+q} \beta_k^{i2} = 1$$

The dW_t^{ck} corresponds to the diffusion driver common to several issuers survival, while the moves of $dW_t^{\epsilon i}$ are specific to the survival of issuer i. When dealing with large pool of exchangeable issuers, symmetry considerations will leads us to grow linearly with the number n of issuers the dimension of the idiosyncratic components space qn, whereas the common component dW_t^{ck} keeps a fixed dimension p.

4.1 Introducing systemic and idiosyncratic hazard rate

The volatility of the survival probability can be decomposed into its systemic and idiosyncratic part:

$$\begin{aligned} v_t^i(T).dW_t^i &= v^c(t,T).dW_t^c + v^{\epsilon i}(t,T).dW_t^{\epsilon i} \\ \frac{dP_t^i(T)}{P_{t^-}^i(T)} &= -v^c(t,T).dW_t^c - v^{\epsilon i}(t,T).dW_t^{\epsilon i} + \tilde{\lambda}_t^i dt - d\tilde{N}_t^i \\ P_t^i(T) &= P_0^i(T)\,\varepsilon\left(-\int_0^t v^c(s,T).dW_s^c\right)\varepsilon\left(-\int_0^t v^{\epsilon i}(s,T).dW_s^{\epsilon i}\right) \\ & \quad \exp\left(\int_0^t \lambda_s^i ds\right) 1_{\{\tau^i > t\}} \end{aligned}$$

we can define systemic and idiosyncratic hazard rates as follow:

Definition: Systemic and Idiosyncratic Hazard Rates
The hazard rate volatility can always be decomposed into systemic and idiosyncratic parts:

$$v_t^i(T).dW_t^i = v^c(t,T).dW_t^c + v^{\epsilon i}(t,T).dW_t^{\epsilon i} \quad (4.1)$$

Special case: Independence of Systemic and Idiosyncratic Hazard Rates

If $v^c(t,T)$ is $\mathcal{F}_t^c := \sigma(W_s^c, s<t)$ measurable, and $v^{\epsilon i}(t,T)$ is $\mathcal{F}_t^{\epsilon i} := \sigma\left(W_s^{\epsilon i}, s<t\right)$ measurable, the evolution of the systemic and idiosyncratic part are independent. We can then split intensity λ_t^i and hazard rates into two independent, systemic and idiosyncratic processes:

$$\begin{aligned} \lambda_t^i &= \lambda_t^c + \lambda_t^{\epsilon i} && (4.2) \\ h^i(t,T) &= h^c(t,T) + h^{\epsilon i}(t,T) && (4.3) \end{aligned}$$

The corresponding hazard rate diffusion is:

$$\begin{aligned} dh^i(t,T) &:= v^c(t,T).\sigma_t^c(T)\,dt + v_t^{\epsilon i}(T).\sigma_t^{\epsilon i}(T)\,dt \\ &\quad + \sigma^c(t,T).dW_t^c + \sigma^{\epsilon i}(t,T).dW_t^{\epsilon i} \\ \sigma^c(t,T) &:= \frac{\partial}{\partial T} v^c(t,T) \\ \sigma^{\epsilon i}(t,T) &:= \frac{\partial}{\partial T} v^{\epsilon i}(t,T) \end{aligned}$$

This corresponds to a popular class of reduced form multi-name models. The systemic and idiosyncratic volatilities independence results in a more tractable model. The affine jump diffusion framework in [7] is one example.

4.2 LHP loss distribution in the independent systemic case

We assume that Large Homogeneous Pools of issuers[13] with identical dynamics exist. If all issuers are in the same pool, we have: $P_0(T) = P_0^i(T)$. We can now factorise the common

[13] A large homogeneous pool model, verifies the assumption of the De Finetti theorem: that an arbitrarily large number of exchangeable issuers can be found.

terms of the survival probabilities occurring in a portfolio loss:

$$
\begin{aligned}
P_t^i(T) &= P_0(T)\,\varepsilon\left(-\int_0^t v^c(s,T).dW_s^c\right) \\
&\quad \varepsilon\left(-\int_0^t v^{\epsilon i}(s,T).dW_s^{\epsilon i}\right)\exp\left(\int_0^t \lambda_s^i ds\right)1_{\{\tau^i>t\}} \\
\sum_{i=1}^n \frac{1}{n}P_t^i(T) &= Q_t(T)\sum_{i=1}^n \frac{1}{n}M_i \\
Q_t(T) :=& \quad P_0(T)\varepsilon\left(-\int_0^t v^c(s,T).dW_s^c\right) \\
M_i := \frac{P_t^i(T)}{Q_t(T)} &= \varepsilon\left(-\int_0^t v^{\epsilon i}(s,T).dW_s^{\epsilon i}\right)\exp\left(\int_0^t \lambda_s^i ds\right)1_{\{\tau^i>t\}}
\end{aligned}
$$

One can show that risk of $\sum_{i=1}^n \frac{1}{n}M_i$ completely diversifies away for a large portfolio:

$$
Var\left(\sum_{i=1}^n \frac{1}{n}M_i\right) = \frac{1}{n^2}\sum_{i=1}^n Var(M_i) + \frac{1}{n^2}\sum_{i,j} Cov(M_i, M_j) \to 0
$$

This is not evident since the M_i include terms that are not independent, we show in appendix B.1 a proof that relies on hypothesis (HH2). This means that the portfolio expected loss variance conditional on $Q_t(T)$ tends to 0, and that $Q_t(T)$ gives the density of large pool losses.

Property: Convergence of the Density of Spot and Forward Loss (Independent Systemic and Idiosyncratic Hazard Rates)

The loss from 0 to T for a homogeneous portfolio with n issuers defined by

$$
L_T := \frac{1}{n}\sum_{i=1}^n 1_{\{\tau^i<T\}}
$$

has its density at t of forward loss to time T converge to:

$$
E(L_T|\mathcal{G}_t)\overrightarrow{n\to\infty}1 - Q_t(T)
$$

with the following definition for the systemic forward loss process $Q_t(T)$:

$$
\begin{aligned}
Q_t(T) &:= P_0(T)\varepsilon\left(-\int_0^t v^c(s,T).dW_s^c\right) &(4.4)\\
&= \exp\left(-\int_0^t \lambda_s^c ds\right)\exp\left(-\int_t^T h^c(t,u)du\right) &(4.5)
\end{aligned}
$$

From formula (4.4), we see that $Q_t(T)$ is a local martingale with diffusion:

$$
\frac{dQ_t(T)}{Q_t(T)} = -v^c(t,T).dW_t^c
$$

This result shows how the density of forward loss is linked to the survival dynamics. *Information on the dynamics of $Q_t(T)$ gives information on the density at t of forward loss to T for a large portfolio. This also gives us information on the forward loss* green function as the density at a time t_1 of the expected loss to T given the expected loss to T at another time t_2 depends on the diffusion of the systemic forward loss process $Q_t(T)$.

This forward loss is starting at time 0, not t, so that defaults that occurred before t are not “forgotten” by this variable, which does not only include the forward losses from t to T, but also sums up the then historical losses from 0 to t. This is a consequence of using unconditional market dynamics of equation (2.10) rather than an equation conditioned on $\tau^i > t$.

4.3 Diversification effects in general

There is a large class of models where volatility is specified as a local function of intensity λ_t^i. In such case the intensity can not be split into a sum of independent, systemic and idiosyncratic processes[14].

While the evolution of hazard rate can no longer be split into the evolution of two independent variables, the systemic intensity and hazard rates can also be redefined by projection on $\mathcal{F}_t^c$, giving the same definition for $Q_t(T)$ as in (4.5):

$$\exp\left(-\int_0^t \lambda_s^c ds\right) := E\left(\exp\left(-\int_0^t \lambda_s^i ds\right)\Big| \mathcal{F}_t^c\right) \tag{4.6}$$

$$\exp\left(-\int_t^T h^c(t,u)du\right) := E\left(\exp\left(-\int_t^T h^i(t,u)du\right)\Big| \mathcal{F}_t^c\right) \tag{4.7}$$

$$Q_t(T) = \exp\left(-\int_0^t \lambda_s^c ds\right)\exp\left(-\int_t^T h^c(t,u)du\right) \tag{4.8}$$

When considering systemic and idiosyncratic decomposition, "idiosyncratic" hazard rates can be defined either as the difference between hazard rate and systemic hazard rate, or as the projection of the hazard rate onto the idiosyncratic filtration. *While the two definitions coincide in the independent systemic and idiosyncratic hazard rate diffusion case, this is not the case in general,* and we face the dilemma of having idiosyncratic hazard rates that can be negative, or systemic components and idiosyncratic components that do not add up to the hazard rate. Fortunately, we do not need the idiosyncratic hazard rates in any of the following.

To identify large homogeneous pool loss behaviour in the general case, we can use the generalised definition of $Q_t(T)$ by (4.9). We show in appendix B.2 that *we essentially have the same diversification property*:

Property: Convergence of the Density of Spot and Forward Losses (General Case)

The loss from 0 to T for a homogeneous portfolio with n issuers defined by

$$L_T := \frac{1}{n}\sum_{i=1}^n 1_{\{\tau^i<T\}}$$

has its forward density given market information up to time t converge to:

$$E(L_T|\mathcal{G}_t)\overrightarrow{n\to\infty} 1 - Q_t(T)$$

with the general definition for the systemic forward loss process $Q_t(T)$:

$$Q_t(T) := E\left(P_t^i(T)\,|\mathcal{F}_t^c\right) \tag{4.9}$$

where the systemic filtration in defined by: $\mathcal{F}_t^c := \sigma(W_s^c, s<t)$. $Q_t(T)$ is a $\mathcal{F}_t^c$ adapted martingale[a]*, its diffusion equation is:*

$$\frac{dQ_t(T)}{Q_t(T)} = -E(v_t^i(T)\,|\mathcal{F}_t^c).dW_t^c \tag{4.10}$$

[a] This is now clear from its general definition(4.9)

We see that the question of finding what the systemic intensity dynamics is for a large pool portfolio remains the same whether volatilities are evolving in independent filtrations $\mathcal{F}_t^c$ and

[14] A correlated lognormal or CIR diffusion of intensity would not be separable, we also want to be able to deal with non separable cases because their risks are different. The structural model diffusion implied by KMV and Credit Metrics for the gaussian copula also belong to this category.

$\mathcal{F}_t^{\epsilon i}$ or not: it boils down to the same of $P_t^i(T)$ projection onto $\mathcal{F}_t^c$. On a practical level, however, the fitting of a single name volatility to that systemic intensity dynamic is no longer a trivial task when volatility is not separated.

4.4 Extension to jump diffusion dynamics

Stepping back, the results we obtained are not a consequence of the volatility specification that was chosen, of its separability, but rather a consequence of the conditional independence properties we assumed.

Now that we explicited the variable and filtration conditional on which defaults are independent, we can add some poisson jumps N_t^k of size $J_t^{ik}(T)$ to hazard rates. The survival dynamics are then:

$$\frac{dP_t^i(T)}{P_{t^-}^i(T)} = -v_t^i(T).dW_t^i + \tilde{\lambda}_t^i dt - d\tilde{N}_t^i + \sum_k E\left(J_P^{ki}(t,T)\right)\mu_t^k dt - J_P^{ki}(t,T)dN_t^k \tag{4.11}$$

with a link between hazard rate jumps and survival jump given by:

$$J_t^{ik}(T) = \frac{\partial}{\partial T} J_P^{ki}(t,T) \tag{4.12}$$

We saw in the previous sections that identifying the common risk source filtration $\mathcal{F}_t^c$ is central to determining large pool behavior. We denote by J_t^k the factor conditional on which hazard rate jump size $J_t^{ik}(T)$ are independent (this is only saying that hazard rate jump sizes have a copula), the filtration used to determine the loss copula factor would then be extended to contain jump times and jump sizes copula factor realisation:

$$\mathcal{F}_t^c = \sigma\left(W_s^c, N_s^k, J_s^k, s < t\right) \tag{4.13}$$

We proposed here to add jumps to the hazard rates dynamic to show how work on CDO pricing with jump-diffusion of intensity such as Duffie and Garleanu [7] can be dealt with[15]. While such a hazard rate jumps can be easily be added under the natural probability, the question of whether the market is still complete and risk neutral price can be obtained using expectations, is beyond most articles dealing with jump diffusion so far. We can only remark this caveat here and point the reader to towards incomplete market literature such as Cont et al. [10] where utility function expliciting risk aversion needs to be provided to calculate a price.

4.5 Link between CDO market prices and dynamics of the common factor

How does market data constrains the forward loss density $Q_t(T)$?

We could find the implied local volatility using the Dupire formula if the market allowed us to extract option information on $C(t,K) = E\left((Q_t(T) - K)^+ |\mathcal{G}_0\right), \forall t, K$. Unfortunately, what we have from the tranche market is not options on the same underlying L_T (coterminal options), rather each option expiry t corresponds to a different underlying tenor L_t, CDO price information is therefore $C(t,K) = E\left((Q_t(t) - K)^+ |\mathcal{G}_0\right), \forall t, K$. This means that the natural underlying diffusion variable to which expected tranche loss surface is linked has its horizon $T = t$ move with the exercise date.

[15] Since the jump size of the survival probability is always lower than 1, the finite sum of poisson processes $\sum_k E\left(J_P^{ki}(t,T)\right)\mu_t^k dt - J_P^{ki}(t,T)dN_t^k$ can be replaced by an integral over the jump size using a Levy measure to deal with infinite activity Levy process rather than brownian motion and jumps only. The notation becomes quite harder as the Levy measure ends up depending on (t,T) in a way that is much clearer for with jumps though.

Property: Convergence of the Density of Spot Loss
The loss from 0 to t for a homogeneous portfolio with n issuers defined by

$$L_t := \frac{1}{n}\sum_{i=1}^{n} 1_{\{\tau^i<t\}}$$

has its density at time t of spot losses converge to:

$$L_t = E(L_t|\mathcal{G}_t)\overrightarrow{n\to\infty}1 - Q_t(t)$$

As $T = t$, we have the following properties for the systemic spot loss process $Q_t(t)$:

$$1 - Q_t\left(t\right) = E\left(L_t \middle| \mathcal{F}_t^c\right) = 1 - \exp\left(-\int_0^t \lambda_s^c ds\right)$$

The loss time horizon $T = t$ is now moving at the same speed as the conditioning filtration. So $Q_t(t)$ is not a martingale, whereas $Q_t(T)$ with a fixed horizon T is one. $Q_t(t)$ is actually a drift only process:

$$\frac{dQ_t\left(t\right)}{Q_t\left(t\right)} = -\lambda_t^c dt \tag{4.14}$$

We see here that $Q_t\left(t\right)$ gives the spot loss density at time t for tranche, but that instead of being a martigale, $Q_t\left(t\right)$ is a *drift only process*, whose drift is the systemic hazard rate.

Indeed, instead of a Dupire like formula, where option prices give information on the expected volatility, tranche option surface gives only information on *local intensity formulas* like:

$$\frac{\partial}{\partial T}C\left(T,K\right) = -E\left(1_{\{Q_T(T)>K\}}Q_T\left(T\right)\lambda_T^c\right) \tag{4.15}$$

CDO market only gives us information on expected intensity conditional on loss level, not on its volatility. It is only by making assumptions about intertemporal loss correlation that we can infer intensity volatilities from tranche prices.

4.6 Dealing with Portfolio Heterogeneity

The previous section concerned itself with large homogeneous portfolio results, we need to see how the previous results can be extended when the variables are not exchangeable. From here on, we will assume that the projection onto $\mathcal{F}_t^c$ is different for every issuer i, and simply affix an index i on the systemic forward loss process $Q_t(T)$, whose definition is otherwise unchanged:

Property: Diversification for Heterogeneous Portfolio
The loss from 0 to t for a heterogeneous portfolio with n issuers defined by

$$L_T := \frac{1}{n}\sum_{i=1}^{n} 1_{\{\tau^i<T\}}$$

has its density at time t of forward losses to T converge to the average of the systemic forward loss process:

$$E(L_T|\mathcal{G}_t) - \left(1 - \frac{1}{n}\sum_{i=1}^{n} Q_t^i(T)\right)\overrightarrow{n\to\infty}0$$

with:

$$Q_t^i\left(T\right) \quad := \quad E\left(P_t^i(T)\middle|\mathcal{F}_t^c\right), \quad \forall i$$

It is easy to check from the diversification results proof in appendix that the variance of $E(L_T|\mathcal{G}_t)$, the portfolio loss seen at t, conditional on all the $Q_t^i\left(T\right)$ is 0.

4.7 Conclusion on diversification effects

At that point, after having transposed HJM model to credit in a previous section, we have, with these notations, calculated the large pool loss distribution. We then have shown that this diversification result holds in the general case for the portfolio loss $E(L_T|\mathcal{G}_t)$, and that the right variable conditional on which we obtain diversification is a $\mathcal{F}_t^c$ *martingale.* This result gives insight into t*he drivers of the forward density at t of expected loss to T*, and even *its Green function* (the density at any t_2 of expected loss to T given the expected loss to T at t_1). Even in the general case when idiosyncratic hazard rates cannot be defined in a satisfying manner, the density of spot and forward losses is still driven by the systemic forward loss process $Q_t(T)$.

We also saw that the spot density at t, obtained by setting the loss horizon $T = t$ is linked to the systemic spot loss process $Q_t(t)$, which is a *predictable process with the systemic intensity as a drift*, and that *CDO market does not give information about intertemporal loss correlation.*

Conceptually, this section highlighted the role projection of the portfolio loss onto the common risk factor filtration $\mathcal{F}_t^c$.

We now proceed to check whether this martingale is a suitable copula factor, and how this variable works out in the case of discrete portfolios.

5 Dynamics and CDO Pricing Copula

This section constructs a natural copula factor for any portfolio, among all the possible ones. It then shows that in the general case, and contrary to common practice, whatever the chosen copula, it cannot be one dimensional. Assuming a one factor copula implies assuming a very restrictive dependency structure among the different issuers in the portfolio.

5.1 Independence is stronger than diversification for forward survivals

The previous section only used the fact that the covariance of the $P_t^i(T)$ conditional on $Q_t^i(T)$ is 0. However, this is not sufficient for the $P_t^i(T)$ to be independent. We show in appendix B.3 that independence is obtained only when higher moments are 0, which results for moments of order r in conditioning on some additional variables $Q_t^{i,r}(T)$, where $Q_t^{i,r=1}(T) = Q_t^i(T)$:

Property: Conditional Independence of the $P_t^i(T)$
Although conditional covariances are 0, the $P_t^i(T)$ are not necessarily independent conditional on all the $Q_t^i(T)$. The $P_t^i(T)$ are independent conditional on:

$$\mathcal{F}_t^q := \sigma\left(Q_t^{i,r}(T), \forall i \in [1,\ldots,n], \forall r \geq 1\right)$$

where

$$Q_t^{i,r}(T) \quad := \quad E\left(\varepsilon\left(-\int_0^t v_s^i(T).dW_s^i\right)^r \exp\left(\int_0^t \lambda_s^i ds\right)^{r-1} \middle| \mathcal{F}_t^c\right)$$

On top of our previous result concerning diversification of large pool losses, we see that there is an infinite set of variables conditional on which we have independence. This means that in general, conditioning on a finite number of $\mathcal{F}_t^c$ variables is enough to get diversification effects (0 conditional variance), but not enough to get conditional independence.

To sum up, conditional on the systemic forward loss processes $Q_t^i(T)$, the forward losses diversify, but are not independent.

5.2 Conditional independence for default indicators: the canonical copula

While we saw that conditioning on the systemic forward loss processes $Q_t^i(T)$ was not enough to obtain independence for $T > t$, the case of the systemic spot loss process with $T = t$ is special[16]. The variable $P_t^i(t) = 1_{\{\tau^i > t\}}$ are actually survival indicators. As those are bernoulli variables that take the values 0 or 1, we show in appendix B.3 that the indicators $1_{\{\tau^i > t\}}$ are independent conditional on the $Q_t^i(t)$ only.

To sum up, when $T > t$, the diversification does not translate into independence, as we only obtain the independence of the $P_t^i(T)$ conditional on the sigma field $\mathcal{F}_t^q$. However, for default indicators, which is what interests us for CDO pricing, the diversification results in the independence of the $1_{\{\tau^i > t\}}$ conditional on the systemic spot loss processes $Q_t^i(t)$.

We are now ready to answer the question whether—once a dynamic is specified—we can always exhibit its copula:

Definition: Canonical Copula
The survival indicators $1_{\{\tau^i > t\}}$ are independent conditional on the $\mathcal{G}_t$ measurable random variable U, which is the n dimensional vector with components:

$$U^i = Q_t^i(t) = E\left(1_{\{\tau^i > t\}} | \mathcal{F}_t^c\right) \tag{5.1}$$

The copula of default indicators is therefore defined by the of the density of the "copula factor" U and the conditional survival probabilities $P(\tau^i > t|U)$.

The conditional independence property makes *CDO pricing possible with the standard methods*. A good property of this factor is its $\mathcal{F}_t^c$ *measurability*, which implies that the factor does not change when adding issuers indistinguishable with the ones already in the portfolio. Using a non $\mathcal{F}_t^c$ measurable conditioning factor, like the $\int_0^t \lambda_s^i ds$, would entail having some idiosyncratic information in the factor, and requires that the factor change when adding issuers. *Indeed, assuming that there are n arbitrary large pools of exchangeable issuers, the U^i would lead to only n conditioning variables.*

Thus, conditional on the systemic spot loss processes $Q_t^i(t)$, the spot losses do not only diversify, but are also independent, which causes the existence of the canonical copula.

Having answered positively that there always is a n dimensional random variable, which is even $\mathcal{F}_t^c$ measurable, conditional on which the $1_{\{\tau^i > t\}}$ are independent, the next question is whether dimensionality can be further be reduced.

5.3 Dynamics are in general incompatible with one factor copula

First, we have to deal with the special case of homogeneous portfolios:

Proposition: One Factor Copula for Homogeneous Portfolio
In an homogeneous portfolio: the canonical copula is actually a one factor copula since all the U^i are identical

More generally, one can see that if the U^i are comonotonic[17], the canonical copula is a one factor copula. These are special cases where the systemic hazard rate of non exchangeable issuers are made to be comonotonic, or where all the common information happen at once.

We now deal with the general case of an heterogeneous portfolio and non degenerate dynamics, where there are at least two issuers, such that U^1 and U^2 are decorrelated. Although the canonical copula defined above has non comonotonic factors in that case, there is a further

[16] Algebraically, the expression $Q_t^{i,r}(t)$ can be simplified, but the explanation using idempotence of bernoulli variables is more simple

[17] Variables X, Y are said to be comonotonic iff $\forall \omega_1, \omega_2, X(\omega_1) \leq X(\omega_2) \iff Y(\omega_1) \leq Y(\omega_2)$, which is equivalent to say that their copula is the Frechet upper bound $\min(u, v)$.

difficulty when dealing with the copula of default indicators. The following property is well known:

Proposition: Infinitely Many Copula Alternatives to the Canonical Copula
There is an infinity of different copula that can accomodate a given joint distribution between discrete variables.

This means that the canonical copula is only one copula compatible with the CDO prices, but there are infinitely many others, with potentially different properties. So even if the canonical copula introduced above has several factors, one may still find a one factor copula that works for a given portfolio.

What one can prove however, is the following:

Proposition: No Admissible One Factor Alternative for Large Portfolios
When two issuers have such dynamics that U^1 and U^2 are not fully correlated, there exist one n from which a portfolio composed of two pools of n identical issuers cannot have a one-factor copula.

We show a formal proof of this in appendix B.4. We see that, though there are cases for almost homogeneous portfolios or very small portfolios where a single factor copula can reproduce all CDO prices, for a sufficiently large heterogeneous portfolio, where dynamics imply significant terminal decorrelation of survival, such a one factor model does not exist.

The intuition underlying the demonstration can be summarised as follow: if there was such a one factor copula with factor U, we introduce L_i the loss of the pool i, then L_i follows the binomial distribution with expectation $p = E(L_i|U)$ a standard deviation of $\frac{p(1-p)}{\sqrt{n}}$, and converges in law to a dirac when n tends to infinity. Considering the first pool of issuers means that U_1 and U will have almost no covariance as n increases while considering the second pool means the same for U_2 and U. Thus, the existence of a one factor copula for any sufficiently big n would constrain U_1 and U_2 to be comonotonic.

This last proof is interesting in that it helps understand that adding exchangeable issuers to a portfolio makes its copula converge towards the canonical copula. This leads us to the following conjecture, on which we elaborate in Appendix B.5:

Conjecture: All Copula Alternatives Converge to the Canonical Copula
When p_i issuers exchangeable with issuers i are added in the portfolio, the copula between the $\sum_{i=1}^{n} p_i$ variables converges towards the canonical copula as $\forall i, p_i \to \infty$.

To sum up, *this subsection shows that all the admissible copulas converge to the canonical copula (composed of the $U^i = Q_t^i(t)$ with one i for each group of exchangeable names) when the exchangeable names pool sizes increase.*

5.4 Conclusion on pricing copula

The De Finetti's theorem already granted us similar results concerning the distribution of a sum of exchangeable bernoulli variables (say $1_{\{\tau^i > T\}}$); what we have explicited here is a link between the De Finetti result conditioning variable and the $\mathcal{G}_t$ survival dynamics, and a generalisation of the De Finetti's theorem when issuers are not exchangeable.

In general, the default indicator copula, which is all we need to price a CDO, will be a n factor copula. The factor dimensionality cannot in theory be reduced to one, unless in very specific cases.

As we can see, the hypothesis that arbitrarily many exchangeable names can be added to a portfolio not only led us to argue that multi-name models should not have contagion and formulate hypothesis (HH2), it also helped us to *better appreciate the applicability of the results concerning the canonical copula.*

6 Expliciting the Underlying One Factor in CDO Models

This section proposes a review of the most popular current models (copula of default times models, copula of default indicators models, markovian loss intensity models, affine jump diffusion intensity models) that have been put forward for their tractability and usually assume one factor copulas. We explicit the underlying dynamics of those models, and show how this one factor assumption constrains their properties and leads to unsatisfactory results.

We know from de Finetti's theorem that a one-factor copula can be defined for large homogeneous portfolio. We have now shown that a portfolio containing several groups of exchangeable issuers can be priced with a copula with as many factors as groups but that the copula factor dimension can not be reduced when the portfolio is sufficiently large in general.

Since most models put forward for CDO pricing correspond to one factor copula model, we will review some of these models and the way the implied factor structure influences their properties. The three classes we review are the gaussian copula models, the local intensity models, and the affine jump diffusion models.

6.1 One-factor gaussian copula

6.1.1 One Factor Copula of Default Times: Li (2000), Schonbucher et al (2001)

The joint distribution of default times is determined by the marginal default time distributions (given by hazard rates) and the default time copula. In this setup, we assume that the copula of default time has one factor V. That is, default indicator for all $T > 0$ are independent conditional on V. Their conditional probability is given by a function g such that determines the default

$$\Pr(\tau^i < T|V) = g(P_0^i(T), V)$$

A dynamic interpretation goes as follow: In the case of a LHP, because of de Finetti's theorem, we know that the factor V can be mapped to the portfolio loss. The fact that V does not depend on the time imply that knowing the proportion of defaults which have happened in any arbitrarily short period in a large pool, and knowing that the defaults are independent given V, one can exactly know V arbitrarily early. So, for a large enough credit universe, V is $\mathcal{F}_{0+}$ measurable, and the *dynamics* for $t > 0$ are given by:

$$\begin{aligned}
\frac{dP_t^i(T)}{P_{t^-}^i(T)} &= \lambda_t^i dt - dN_t^i \\
v_t^i(T) &= 0 \\
\lambda_T^i &= h^i(0,T)\frac{P_0^i(T)}{g(P_0^i(T),V)}\frac{\partial g}{\partial p}(P_0^i(T),V)
\end{aligned}$$

For instance, in the gaussian copula case[18]

$$\begin{aligned}
g(P_0^i(T),V) &= \Phi\left(\frac{\Phi^{-1}(P_0^i(T)) - \sqrt{\rho}V}{\sqrt{1-\rho}}\right) \\
\lambda_T^i &= h^i(0,T)\frac{\Phi(u_i)}{\varphi(u_i)}\frac{1}{\sqrt{1-\rho}}\frac{\varphi\left(\frac{u_i-\sqrt{\rho}V}{\sqrt{1-\rho}}\right)}{\Phi\left(\frac{u_i-\sqrt{\rho}V}{\sqrt{1-\rho}}\right)} \\
u_i :=& \quad \Phi^{-1}(P_0^i(T))
\end{aligned}$$

While the model's input seem innocuous, as we enter single name hazard rates (marginals), and a default time copula which is the modeler's choice, this setup results in degenerated implied dynamics in the LHP case: all default intensities realize all their randomness at time 0^+, then are completely deterministic. This model has been qualified as a *static* one, in the sense that

[18] We use the following notations for the normal density and cumulative function: $\varphi(u) := \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}u^2}$, $\Phi(u) := \int_{-\infty}^{u}\varphi(z)dz$

intensities are deterministic for $t > 0$, and though a copula of default times exists necessarily, since default times are bound to have a joint distribution, this powerful tool coming from actuarial finance seems better fitted to calculating expectations, rather than give a risk neutral price that corresponds to a realistic replication strategies. Under this specification all the credit risk uncertainty concerning the evolution of intensities is resolved from the start, and replication, once V is known, only involves default hedges.

If the credit universe is composed of n issuers, not of an infinity of issuers, at $t = 0^+$ we have no information yet on the distribution of V. But V is still non time dependent. In that case, even if all is not known at $t = 0^+$, the arriving information flow does not represent the evolution of the state of the economy, but a better knowledge of the static variable V. The speed of arrival of the information about the factor V depends on the number of issuers in the portfolio : the bigger the number of issuers, the quicker V is known, and the more the model degenerates into the deterministic intensity model that is the large pool case. This case was given a detailed technical study in the Schönbucher [8] paper, where it is shown that under a copula of default time assumption, the dynamics of intensities depend on how many issuers are being observed in the credit universe.

So it seems curious at first that such a degenerate model comes from such generic assumption. Upon closer inspection, the assumption that there is a one factor copula of default time is at variance with the copula coming from non degenerated dynamics.

To sum up, the large pool behavior of this model exposes the fact that all the uncertainty about intensities is realised at once. It leads to far too high an intertemporal loss correlation.

6.1.2 Gaussian Copula of Default Indicators: Vasicek (89), Gupton (97)

Just as one can exhibit many dynamics that result in a terminal lognormal distribution for an asset price, there are many dynamics that result in a gaussian copula of default indicator. The probabilistic interpretation of CDO pricing in terms of default time copula is comparatively new (Li [6]). While this approach had the merit of showing how to build all CDO pricing model that are statically consistent, its interpretation in terms of dynamics have drawn criticism.

For this reason, we will also present the probabilistic interpretation implied by anterior models that established gaussian copula CDO pricing as the market standard: with KMV (Vasicek [3]) and Credit Metrics (Gupton [4]), and that had a dynamic interpretation and replication strategy consistent with current market practice. As per Merton [1] earlier structural model approach, the default is modeled as as a stochastic process X_t^i ending up below a barrier at the credit observation date T.

$$\{\tau^i = T\} \quad <=> \quad \{X_T^i \leq 0\} \tag{6.1}$$
$$dX_t^i \quad = \quad \sigma_t . dW_t^i \tag{6.2}$$

One can check that the T default indicators have a gaussian copula with correlation matrix ρ_{ij} where:

$$dW_t^i . dW_t^j = \rho_{ij} dt$$

which effectively construct a one factor gaussian copula on X_T^i. The σ_t have to be identical for all issuers for the copula to be a one–factor copula; otherwise the integrated brownians would decorrelate.

Survival probability is given by the probability of staying above the barrier:

$$P_t^i(T) \quad = \quad \Pr(\tau^i > T | \mathcal{F}_t) = \Phi\left(\frac{X_t^i}{\varsigma(t,T)}\right)$$
$$\varsigma(t,T) \quad := \quad \sqrt{\int_t^T \sigma_s^2 ds}$$

which, using Ito's lemma, gives us the model dynamics:

$$\begin{aligned}\frac{dP_t^i(T)}{P_{t^-}^i(T)} &= -v_t^i(T).dW_t^i \\ v_t^i(T) &= \theta(t,T)\frac{\varphi\circ\Phi^{-1}(P_t^i(T))}{P_t^i(T)} \\ \lambda_t^i &= 0 \\ \theta(t,T) &:= \frac{\sigma_t}{\sqrt{\int_t^T \sigma_s^2 ds}}\end{aligned}$$

One should note that the dt terms obtained from Ito's formula cancel each other, which is expected since $P_t^i(T)$, begin a $\mathcal{F}_t$ conditional expectation, is a martingale. The expression $\frac{\varphi\circ\Phi^{-1}(X)}{X}$ has the asymptotic behavior of $\Phi^{-1}(X)$ if X gets near 0 or 1, which means that volatility explodes when nearing these points.

This time, V is only known once $\mathcal{F}_t$ is revealed, and replication involves delta hedging perturbations of the $P_t^i(T)$.

Remark: In a traditional Black-Scholes scheme, an option with maturity T has different prices at times $t_1$and t_2 even if the underlying spot are the same $S_{t_1} = S_{t_2}$, because time value depends explicitly of time to expiry $T-t$. The Black-Scholes option price is a martingale because the gamma positive component of P/L is expected to offset time decay.

In the gaussian copula model, the price is a function of the default probability, not time. Option prices are still martingales, and do not have time dependence, the time value in this model is 0, and the gamma is 0.

The implicit variance of the payoff is completely embedded in the survival probabilities, with a formula such as:

$$P_t^i(T) = \Phi\left(\frac{X_t^i}{\varsigma(t,T)}\right) \tag{6.3}$$

which contains both the spot information and the remaining time information. This is unlike intensity models where λ_t^i diffuses with a finite volatility, leading to a variance of $\int_t^T \lambda_u^i du$ that tends to 0 as $t \to T$. What we see here instead is that survival volatility explodes when $P_t^i(T)$ tends to 0.

To sum up, this interpretation is more intuitive and leads to a replication argument consistent with current practice. However, using a single factor leads to higher correlation between groups of very risky issuers and groups of very low risk issuers. Another characteristic of this model is that the variance information on survival is assumed to be embedded in the level at which survival probability is, not the time $T-t$ remaining until the default is observed.

6.2 Markovian Loss Intensity Models

We define the portfolio intensity λ_t for a portfolio with notionals w_i and recoveries R_i as the compensator of the loss, by:

$$N_t := \sum_{i=1}^{n} w_i(1-R_i)1_{\{\tau^i>t\}} \tag{6.4}$$

$$\lambda_t^i 1_{\{\tau^i>t^-\}}dt := -E(d1_{\{\tau^i>t\}}|\mathcal{G}_t) \tag{6.5}$$

$$\lambda_t = \frac{1}{N_{t^-}}\sum_{i}^{n} w_i(1-R_i)1_{\{\tau^i>t^-\}}\lambda_t^i \tag{6.6}$$

The portfolio intensity λ_t is $\mathcal{G}_t$ adapted. Since we lose information when projecting on $\mathcal{F}_t$ irrespective of whether λ_t^i are $\mathcal{F}_t$ adapted or not, portfolio intensity λ_t, does not lend itself to the definition of a $\mathcal{F}_t$ background intensity. It is a $\mathcal{G}_t$ specific concept, and we no longer require the assumption of conditional independence in this section, as we work on $\mathcal{G}_t$.

We recall that tranche prices determine the portfolio intensity conditional on the loss amount:

$$\begin{aligned}
\frac{d}{dK}(L_t - K)^+ &= 1_{\{L_t > K\}} \\
\frac{d^2}{dKdt}E((L_t - K)^+) &= E(1_{\{L_t = K\}}dL_t) \\
&= E(dL_t|L_t = K)f(K) \\
f(K) &:= \frac{d^2}{dK^2}E((L_t - K)^+) \\
\frac{d^2}{dKdt}E((L_t - K)^+) &= (1-K)E(\lambda_t|L_t = K)f(K)
\end{aligned}$$

Given that portfolio intensity is defined by tranche market prices, a very simple model that guarantees correct calibration to market prices is to have a model were the portfolio intensity is a deterministic function of L_t.

For a finite portfolio, such a property means that we have a pure contagion model: the intensities filtration can not contain any other risk source than the default times of issuers in the portfolio.

For a large homogeneous pool portfolio however, the large pool rate of default can be interpreted as an exogenous variable that determines the loss level. So, despite the fact the finite portfolio case is a contagion model, it tends, when the portfolio is large enough, towards a model with a single variable $Q_t(t)$ that describes the state of the pool, whose intensity we showed earlier is λ_t^c. But here again, due to diversification effect, using local intensity means that knowing the value of the loss L_t at some $t > 0$ determines the intensities for all times $T > 0$. Thus, when the number of issuers increases, it degenerates to a model where all the randomness is resolved at $t = 0^+$, like the one-factor copula between default times.

To sum up, markovian loss intensity models are contagion models that degenerate in large pool to a one factor default time copula model calibrated to tranche prices.

6.3 Affine Jump Diffusion Intensity: Duffie and Garleanu

We review the models that are most consistent with the framework proposed in this paper, and that are built from the bottom up with non degenerate dynamics.

Each name intensity process λ_t^i is modeled as a linear combination of a systemic intensity λ_t and an independent idiosyncratic intensity. The idiosyncratic intensity dynamic does not impact the copula of default indicators, so only the model of the systemic intensity needs to be specified. An affine jump diffusion model is used:

$$\lambda_t^i = \beta_i \lambda_t + \lambda_t^{\epsilon i} \tag{6.7}$$

$$d\lambda_t = (\mu_t - k\lambda_t)dt + \sigma\sqrt{\lambda(t)}dW_t^i + dJ_i(t) \tag{6.8}$$

The natural copula factors in this model are

$$E(1_{\{\tau^i > t\}}|\mathcal{F}_t^c) = \exp\left(-\int_0^t \beta_i \lambda_s ds\right) \tag{6.9}$$

$$= Q_t(t)^{\beta_i} \tag{6.10}$$

In this model, we see that the default copula reduces to one factor if the β_i are time independent. Choosing non constant β poses question of arbitrage freeness[19] and leads to multifactor model unless the β_i are proportional to each other. The good property of this model is that, although it is a one factor default indicator copula, its factor has a diffusive behavior leading to a more realistic modeling of intertemporal loss.

At this stage, we recall a result concerning the link between default indicator copula versus spread correlation: spread correlation is necessary but far from sufficient to induce correlation

[19] The β_t needs to be increasing to guarantee an increasing compensator on all trajectories.

between defaults: if the spreads have no volatility the defaults are independent, because defaults are independent conditional on intensity. A spread volatility of 0 imply a default correlation of 0; there is no default indicator correlation without spread volatility. Thus, to simulate a gaussian copula default correlation of 90%, we have to have not only very high spread correlations but also very high spread volatilities.

For this model, the choice of a one-factor dynamic led us to use a proportional participation of every name to the systemic factor. We see that if there is some very tight name in the portfolio, so that β_i is tiny, this name spread will have a very low volatility, meaning that this name is almost independent from the others.

To sum up, this model has the most realistic intertemporal loss correlation, as it corresponds to a copula factor with non degenerate dynamics, however, the modeling choice that was made to ensure it can be implemented as a one factor copula model leads to deltas that are too low on senior tranches for tight issuers.

7 Conclusion

7.1 Unifying the static and dynamic approaches

By changing the interpretative framework for copula, and presenting the copula as a phenomenological device used to "measure" dependence rather than a causal driver of default dependence, the replication argument in a generic framework becomes clear and consistent with derivative pricing theory. This contributes to bring credit derivative valuation at par with other asset classes models, in terms of theoretical foundation, although questions such as the number of underlyings, and what market information is available to calibrate models means that in practice, there will always be issues specific to CDOs pricing.

Up to now, the prevalent market practice is to use a static copula approach with market implied probability for pricing and hedging, which means that the link between the static (copulas) approach and the dynamic (intensities) approach needed to be understood. The former attempts at unifying both approaches suffered many practical and theoretical drawbacks.

We proposed here a unified view, where the default dependence is caused solely by the risk neutral dynamics of default probabilities in the market filtration, and the copula is seen only as a consequence of these dynamics.

In this article, we directly modelled intensities in the market filtration and obtained the General Credit Term Structure Dynamics (2.8) and (2.10), then derived the factor default indicator copula implied by survival dynamics, (4.2) and (4.10) and (4.14), and explicited the random variables conditional on which defaults are independent in section (5.2).

In general, the default indicator copula, which is all we need to price a CDO, will be a n factor copula. While this is often done in practice to make pricing more tractable, the factor dimensionality cannot in theory be reduced to one, unless in very specific cases : see section (5.3) .

7.2 Perspective on Financial Modeling

Taking a step back from the issues at hand with credit modeling, one can distinguish two phases in the modern development of derivatives pricing.

A first phase, where pioneers find simple models and have an enduring legacy: the case in point would be options on stocks, which were blessed with the early discovery of the Black Scholes model[20]. But this phase continued well into the 80s as we still observed remarkable creativity in rates modeling with important and tractable ideas put forward by Vasicek or CIR.

A second phase, which took place from the second part of the 80s, and well into the 90s, was to develop a better understanding of why these models work, or how they can be improved. To this category, we assign some pragmatic changes, such as the addition by Hull-White of time varying parameters to the Vasicek model. There are also theoretical breakthrough that apply

[20]Model whose versatility and robustness came to be understood much later, over years of practice

to all models of a certain category and improve the understanding of the field: the change of numeraire technique was a fundamental tool for extending the applicability of previous models. Another example, the HJM framework, gave a unifying framework to compare rates model, and established the fact that the drift of the short rate was a consequence of the forward rate volatility specification (mean reversion is a consequence of exponentially decreasing forward volatility).

While the first phase is marked by creative genius, and candid choices, some of which turn out to be judicious, in the second phase, new modeling ideas can only be the result of a conscious understanding of the properties or limitations of existing models, and a deliberate attempt to overcome them.

Due to the rather late development of credit derivatives market in the late 90s, and the scarlet letter of having used such actuarial tools as copula in its infancy, Credit Derivatives Modeling is often seen as still in phase one, so that many are hoping for some pioneering work that would produce a simple, tractable "dynamic" model for credit that would fit market prices, and that this model would turn out to have the "good[21]" properties. As this field of research is more than 10 years old, we decided to position ourselves deliberately in the second category, and expound on why existing models work and what needs to be improved in them.

7.3 Consequences on existing models

We have analysed why some of the CDO pricing models proposed so far tend to imply unsatisfactory properties. The first limitation we noticed is the classical one factor copula assumption. This approximation of the n factor copula implied by the dynamics is perfect when the portfolio is homogeneous, but it breaks down when the portfolio components are very heterogeneous. Nevertheless, the use of a copula with a few more factors will be better suited, and compatible with market practice.

In copula models, we notice that the use of default indicators copulas is much more general than default time copulas, the latter structurally imply degenerate intertemporal correlation. Still, the use of a one factor copula of indicators leads to a too high correlation between tight and wide spreads. The markovian loss models, which in the finite portfolio case can only be interpreted as contagion models, have a limit in the large pool case that also is a one factor default time copula. And finally, constraining an intensity diffusion model so as to reduce it to a one factor copula by forcing a proportional relationship on systemic intensities leads to the same problem as with any one factor copula of default indicator.

Thus, the proper description of the survival probabilities in the market filtration, where they become martingales, paved the way to the unification between "static" credit models and "dynamic" credit models. It shows how current credit modelling can be properly connected to the derivatives literature and concepts. It creates a new framework that will help understanding the properties and limitations of credit models.

We have left some conjectures in the article, examples of questions still open for us, as we hope this new framework will lit new interest and comprehension.

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A Appendix on Model Setup

A.1 Definition and Properties of the background filtration

In credit literature, the usual constructions require an arbitrary choice of subfiltration, among the many subfiltrations of $\mathcal{G}_t$ to choose from. The question as one accumulated such assumptions, is whether there actually existed a subfiltration that verify all the hypothesis required, and how the intensities for admissible background filtrations may differ. Up to now, we could not find realistic cases where the (HH1) hypothesis was not verified.

While the subfiltration construction proposed in this article is original, its properties make it a "good choice" for the usual credit setup. The (HH1) hypothesis is verified for instance, if we assume that τ^i is the first jump time of a poisson process counter[22] N_t^i with intensity λ_t^i, in which case we have:

$$\tilde{\lambda}_t^i = \lambda_t^i 1_{\{\tau^i > t\}}$$

but is in fact verified for a much larger class of $\mathcal{G}_t$ stopping times; not only does it coincide with the constructive approach where default times are events happening randomly following default intensitites trajectories that have been drawn before, its merit to us is that it properly isolates default time information, as other default times τ^j realisations do not change the distribution of τ^i given a path of background intensity $\lambda_t^i, t > 0$.

We show here that the background filtration defined as verifying the (HH1) hypothesis verify the properties that usually arbitrarily imposed on subfiltrations in credit.

A.1.1 Definition of the Default Realisation Marker $\tilde{\theta}^i$

The default intensity $\tilde{\lambda}_t^i(\omega)$ is the pathwise derivative of the default compensator with respect to t, as the latter is monotonic, its derivative is well defined almost everywhere and has at most a countable set of singularities (diracs).

[22] A poisson counter N_t^i can continue to jump after reaching 1 whereas the default indicator $1_{\{\tau^i < t\}}$ can not

The usual definition for $\tilde{\theta}^i$ in well behaved intensity models (models with finite intensity and almost sure default given infinite time) is:

$$\tilde{\theta}^i = \exp\left(-\int_0^{\tau^i} \tilde{\lambda}_t^i dt\right) \tag{A.1}$$

We need to deal with singularities and the possibility of a non sure default at infinity to generalise this definition. The 0 measure set on which the derivative is not defined does not change anything to $\tilde{\theta}^i$ which only deals with integrated intensity.

When there are diracs in the intensity, using (2.2) as a general definition would led to a $\tilde{\theta}^i$ with diracs. Instead, what we can do is introduce an uniform variable U_i whose measure can be constructed to be independent from P, we can then define:

$$\tilde{\theta}^i = \exp\left(-\int_0^{\tau^{i-}} \tilde{\lambda}_t^i dt + U_i \int_{\tau^{i-}}^{\tau^i} \tilde{\lambda}_t^i dt\right) \tag{A.2}$$

When the default is not sure at infinity, $p_\infty = \exp\left(-\int_0^\infty \tilde{\lambda}_s^i ds\right) > 0$, the standard definition would also lead to a lumpy $\tilde{\theta}^i$, with a lump at p_∞, and we can use we can use the same uniform variable to transform the lump density into a uniform one. To keep notations simple, say we add a dirac at infinity, with the convention that $\tilde{\lambda}_\infty^i = p\infty$ and that $\tau^i = +\infty^-$ when default never occurs.

A.1.2 Existence of $\mathcal{G}_t^{-i}$, no unicity

We can build chains of bigger and bigger subfiltrations of $\mathcal{G}_t$ by adding sets and checking that the generated filtration measurable variables are independent of $\tilde{\theta}^i$. Each of this chain is majorated by $\mathcal{G}_t$, thus a maximal element $\mathcal{G}_t^{-i}$ exists thanks to the Zorn lemma. The maximal element of a chain is attained when we cannot add a new set while staying independent of $\tilde{\theta}^i$.

Of course, in general, there is no unicity of the maximal subfiltration $\mathcal{G}_t^{-i}$ created that way, because each path is different and might end up with a different filtration. Our hypothesis (HH1) states that it is possible to find one of them which is big enough to reproduce the behavior of the default intensity before the default.

There might also be cases where several different maximal subfiltrations verify (HH1). In such cases, they will provide different background intensities, intensities which are all equal before the default, but will slightly differ after. Intuitively, up to the time of default, the information accumulated on the default intensity is limited, thus its law is not perfectly known : small modifications of the parameters of the law might create slightly different prolongations that are still independent from the default realisation (from which the default time can not be distinguished).

A.1.3 Background intensity is $\mathcal{G}_t^{-i}$ adapted

The $\mathcal{G}_t^{-i}$ measurability is assured by the right side member:

$$\lambda_t^i := E(\tilde{\lambda}_t^i | \mathcal{G}_t^{-i}, \tau^i > t) = \frac{E(\tilde{\lambda}_t^i | \mathcal{G}_t^{-i})}{P(\tau^i > t | \mathcal{G}_t^{-i})}$$

We see that the background intensity is well defined on any path ω until such time as default becomes almost sure, that is until such point where the default compensator is infinite and there is no point further diffusing an intensity.

A.1.4 Decreasing background survival probability with t:

This is a common condition for background filtrations. Basically, moving the background filtration forward in time does not bring additional insight into past defaults. The fact that λ_t^i

coincides with $\tilde{\lambda}^i_t$ and $\tilde{\theta}^i$ has uniform distribution $U(0,1)$, and is independent of the background filtration are all used for this demonstration:

$$\begin{aligned} \tau^i > t \quad &<=> \quad \tilde{\theta}^i < \exp(-\int_0^t \lambda^i_s ds) \\ \Pr(\tau > t|\mathcal{G}^{-i}_t) \quad &= \quad \exp(-\int_0^t \lambda^i_s ds) \end{aligned}$$

A.1.5 Martingales for the background filtration are $\mathcal{G}_t$ martingales

This property, which is a usual condition (see Bielecki, Jeanblanc, Rutkowski [11]) for subfiltrations is verified here. As shown by Bielecki et al., it can be recast in a more technical form: that $\mathcal{G}^{-i}_\infty$ is independent of $\mathcal{G}_t$ conditional on $\mathcal{G}^{-i}_t$. As the background filtration is defined by its independence to $\tilde{\theta}^i$, this can be seen as a consequence of the preservation of independence properties by projection. Another formulation for it states that martingales w.r. to the background filtration are still martingales for the market filtration. For any variable X, we define:

$$\begin{aligned} Y :=& \quad E(X|\mathcal{G}^{-i}_\infty) \\ Y \quad \perp& \quad \tilde{\theta}^i \\ Z :=& \quad E(Y|\mathcal{G}_t) \\ Z \quad \perp& \quad \tilde{\theta}^i \end{aligned}$$

Now, Z is $\mathcal{G}_t$ measurable and independent of $\tilde{\theta}^i$, which means that it is also $\mathcal{G}^{-i}_t$ measurable. Hence:

$$\begin{aligned} E(Y|\mathcal{G}_t) &= E(Y|\mathcal{G}^{-i}_t) \\ E(E(X|\mathcal{G}^{-i}_\infty)|\mathcal{G}_t) &= E(E(X|\mathcal{G}^{-i}_\infty)|\mathcal{G}^{-i}_t) \\ E(E(X|\mathcal{G}^{-i}_\infty)|\mathcal{G}_t) &= E(X|\mathcal{G}^{-i}_t) \\ E(E(X|\mathcal{G}^{-i}_\infty)|\mathcal{G}_t, \mathcal{G}^{-i}_t) &= E(X|\mathcal{G}^{-i}_t) \end{aligned}$$

We therefore get the following result concerning nested expectation, which is characteristic of independence:

$$\begin{aligned} \forall X, E(E(X|\mathcal{G}^{-i}_\infty)|\mathcal{G}_t) \quad &= \quad E(X) \quad \text{cond. } \mathcal{G}^{-i}_t \\ \mathcal{G}^{-i}_\infty \quad &\perp \quad \mathcal{G}_t \text{ cond. } \mathcal{G}^{-i}_t \end{aligned}$$

This property of subfiltration $\mathcal{G}^i_t$ is generally referred to as "hypothesis H" in the literature.

A.1.6 On Kusuoka's remark concerning martingality from backgound filtrations

We have not been specific so far about whether our definition of $\mathcal{G}^{-i}_t$ filtrations for (HH1) is invariant by change of measure. As Kusuoka [5] pointed out, martingality properties for subfiltration such as the (H) hypothesis are not necessarily invariant by change of measure. So the question is: is the (HH1) hypothesis invariant by change of measure?

Kusuoka's example is as follow: using two independent exponential variables τ^1, τ^2 with constant arrival intensity, two filtrations are constructed $\mathcal{F}_t := \sigma(\min(\tau^1, t))$ and $\mathcal{G}_t$ which is generated by the two arrival times: $\mathcal{G}_t := \sigma(\min(\tau^1, t), \min(\tau^2, t))$. By virtue of the independence of the two variables, $\mathcal{F}_t$ verifies by construction the (H) hypothesis ($\mathcal{F}_\infty \perp \mathcal{G}_t$ cond. $\mathcal{F}_t$) After that, a change of measure is constructed under which the variable τ^1 has a different $\mathcal{G}_t$ intensity, from its $\mathcal{F}_t$ intensity, which means that the hypothesis (H) no longer holds under this measure.

This example illustrates the general fact that independence is a measure-related concept, and more specifically, that making a measure change whose Radon Nikodym derivative is not $\mathcal{F}_t$ adapted can introduce new information, which can lead to a loss of martingality.

At that point, our definition of the default realisation marker $\tilde{\theta}^i$ depends on the default intensities $\tilde{\lambda}^i_t$ defined by $\tilde{\lambda}^i_t dt := E(d\tilde{N}^i_t|\mathcal{G}_t)$ which is thus measure-related. We need (HH1) to hold in the measure under which the survival probabilities are computed. In the context of our initial model setup, this would be naively be interpreted as the natural probability, however, we will see in section A.3 that our equations, if they are to be used for derivatives pricing are meant to hold under the risk neutral probability Q_T.

In this case, we could directly state that (HH1) holds under Q_T irrespective of whether it holds under P, since the natural probability is less of interest to us.

An alternative is to further explicit the assumption under which the Radon Nikodym derivative for the change of measure from P to Q_T is adapted to the subfiltration $\mathcal{G}_t^{-i}$, which means that (H) hypothesis holds for both measures. Interestingly this property has a plausible interpretation in this setup: the market price of default risk η^i for name i needs to be independent of the default realisation $\tilde{\theta}^i$ (the market price of risk can still depend on the overall level of risk, on overall liquidity...).

This has the merit of translating in intuitive terms what the mathematical hypothesis means, and we see that when the market price of risk is indeed uncorrelated with default realisations, hypothesis (H) and (HH1) holding under pricing measure is equivalent to those hypothesis holding under natural probability measure.

A.1.7 Link between market survival probability and background intensity:

The idea here is to generalise the definition of the variable $\tilde{\theta}^i$ which has uniform distribution $U(0,1)$. Conditionally on $\tau > t$, one can define

$$\begin{aligned} \tilde{\theta}_t^i := & \exp(-\int_t^{\tau_i} \tilde{\lambda}_u^i du) \\ = & \tilde{\theta}_0^i \exp(\int_0^t \tilde{\lambda}_s^i ds) \end{aligned}$$

The hypothesis (HH1) means that $\tilde{\theta}^i = \tilde{\theta}_0^i \perp \lambda_u^i \forall u$, therefore

$$\tilde{\theta}_t^i \perp \lambda_u \text{ cond. } \tau^i > t, \lambda_s^i, s < t$$

As conditional on $\tau^i > t$, $\tilde{\theta}_t^i$ is $U(0,1)$, we have

$$\begin{aligned} \tau^i > T \quad <=> \quad & \tilde{\theta}_t^i < \exp(-\int_t^T \lambda_s^i ds) \\ \Pr(\tau^i > T | \mathcal{G}_t, \tau > t) \quad = \quad & E(\exp(-\int_t^T \lambda_u^i du) | \mathcal{G}_t) \\ \Pr(\tau^i > T | \mathcal{G}_t) \quad = \quad & 1_{\{\tau^i > t\}} E(\exp(-\int_t^T \lambda_u^i du) | \mathcal{G}_t) \end{aligned}$$

A.2 Conditional Independence

Given the definition of the $\tilde{\theta}^i := \exp(-\int_0^{\tau^i} \lambda_u^i du)$, we see that their independence guarantees that the default indicators $1_{\{\tau^i > t\}}$ are independent conditional on the trajectory of the background intensities $\lambda_s^i, s < t$.

A.3 Risk Neutral Dynamics

So far, we have not been specific about the probability measure under which the survival probability is evaluated. We now proceed to clarify this notion. In a nutshell, we explain why the relevant pricing measure is the zero-coupon bond measure.

We shall see how eminently the intensity λ_t^i is a concept dependent on probability measure: it is different under natural and risk neutral probability, and changes if the numeraire can jump on default.

We denote the money market account β_t, and the T maturity zero coupon bond $B(t,T)$, the short rate r_t and the forward rate $f(t,T)$. They are all $\mathcal{G}_t$ adapted.

$$\beta_t := \exp\left(\int_0^t r_s ds\right) \quad (A.3)$$

$$B(t,T) := \exp\left(-\int_t^T f(t,u)du\right) \quad (A.4)$$

The money market account is called a risk free asset because it is predictable: $d\ln\beta_t = r_t dt$.

Hypothesis (H3): Arb Free Complete Market : *there is exist a unique measure Q_β such as asset prices are local martingales using the risk free numeraire*

A remark concerning the difference between risk neutral and natural probability intensity: in a complete arbitrage free market, there is a unique market price of risk vector θ_t whose components correspond to the remuneration of any exposure to the risk source of dM_t, (M_t being the semimartingales that generate $\mathcal{G}_t$). For instance, when a process is driven by a P brownian motion W_t^P, the market prices it as if it was driven by a process drifting by θ_t,with $dW_t^Q = dW_t^P - \theta_t dt$, and there is no way to monetize the drift of P and Q without taking risk. For Cox process $M_t = N_t^i$, the natural intensity is λ_t^{iP} and again, there is a market price of risk η_t^i. Change of measure on poisson processes are defined by keeping the same realisations $N_t^i(\omega)$ but changing their intensity $\lambda_t^{iP} = \lambda_t^{iQ} + \eta_t^i$. Again, if instantaneous protection can be arranged at a given price λ_t^{iQ} and all assets dependent on it are priced consistently, there no way to monetize the fact that the natural probability is λ_t^{iP} without taking on default risk.

Another important risk neutral measure is the $B(t,T)$ numeraire measure Q_T. This measure is equivalent to Q_β and can be defined by its Radon-Nykodim derivative:

$$\frac{dQ_T}{dQ_\beta} = \frac{\beta_t}{\beta_T}\frac{B(T,T)}{B(t,T)} = \exp\left(-\int_t^T r_u - f(t,u)du\right) \quad (A.5)$$

To show that intensity changes when numeraire is assumed to jump on default, we assume that the bond prices jump by an amount $J^i(t,T)$ upon the occurrence of increments dN_t^i. The forward jump diffusion under Q_β is given by:

$$\begin{aligned} df(t,T) &= \sigma^f(t,T)\rho v^f(t,T)dt + \sigma^f(t,T).dW_t \\ &- \sum_{i=1}^n E(J^{if}(t,T)(J^i(t,T)-1))dt + J^{if}(t,T)dN_t^i \end{aligned} \quad (A.6)$$

$$v^f(t,T) := \int_t^T \sigma^f(t,u)du \quad (A.7)$$

$$\ln 1 + J^i(t,T) := \int_t^T J^{if}(t,u)du \quad (A.8)$$

Where $\sigma^f(t,T)$ is the forward volatility, which is a $\mathcal{G}_t$ adapted vector. The only constraint is that W_t and W_t^i are vectors expressed on a same base of brownians, and that the components on this base have correlation ρ. Changing measure from Q_β to Q_T involves the following change in brownians and intensity:

$$dW_t^{Q_T} = dW_t^{Q_\beta} + \rho v^f(t,T)dt \quad (A.9)$$

$$\lambda_t^{iQ_T} = E(J^i(t,T))\lambda_t^{iQ_\beta} \quad (A.10)$$

Our goal in introducing the jumps on default $J^i(t,T)$ was to show that risk neutral intensity of T survival probability depends on T in the most general case. The measure Q_T only differs from Q_β for credit when the credit driver W_t^i is correlated with the evolution of the interest rates, or when the numeraire jumps on default. In the following, we only use Q_T intensities and do not need to assume independence of rates and W_t^i. This is the natural measure to study the evolution the dynamics of the traded survival probability, since the defaultable T maturity bond expressed in unit of $B(t,T)$ will be a martingale under this measure:

$$P_t^i(T) := E^{Q_T}\left(1_{\{\tau^i>T\}}\middle|\,\mathcal{G}_t\right) \tag{A.11}$$

One can see what minimal set of instruments is required to complete the market. Let's write the Ito formula for a non linear payoff f:

$$\begin{aligned} V(t,P_t^i(T)) &:= E^{Q_T}\left(f(P_t^i(T))|\mathcal{G}_t\right) && \text{(A.12)}\\ dV &= \frac{\partial V}{\partial t}dt + \frac{1}{2}\frac{\partial^2 V}{\partial P^2}\sum_{k,l} v^{ik}(t,T)\rho^{kl}v^{il}(t,T)dt \\ &+ \frac{\partial V}{\partial P}dP \\ &+ \left(\frac{\partial V}{\partial P}P - V\right)dN_t^i && \text{(A.13)} \end{aligned}$$

To be able to hedge such a non linear product, we need to be able to trade the T maturity defaultable zero coupon bond (which gives the risk neutral probability $P_t^i(T)$) and instantaneous protection, exchanging dN_t^i for λ_t^i at T. So completeness for such instruments is verified by having only these two instruments in addition to the bond $B(t,T)$. If the rates do not jump on default, which is the case if they are adapted to the background filtration, the market price of instantaneous protection paid at any T will be the same λ_t^i, and this instrument can be replicated using instantaneous CDS paid at its natural time t.

A.4 A deeper look at the Survival Dynamics

Now that we introduced further notations for interest rate assets, it is interesting to compare the results we obtained for survival probabilities with those available in rates for zero coupon bonds. Arguably, these differences with the usual rates setup are rather perfunctory, understanding them will help better understand some technical aspects of term structure diffusions.

A.4.1 Removing the conditioning diffusion to survival results in martingality

A first remark concerns the conditioning to survival. As mentioned before, credit literature usually shows the diffusion of survival to T conditional on survival to t, its expression is very similar to the value at t of a T maturing bond:

$$\begin{aligned} E(P_t^i(T)|\tau^i > t) &= \exp\left(-\int_t^T h^i(t,u)du\right) \\ B(t,T) &= \exp\left(-\int_t^T f(t,u)du\right) \end{aligned}$$

As mentioned earlier, both formulas are very similar, and neither are martingales in the usual HJM measure:

$$\begin{aligned} \frac{dB(t,T)}{B(t,T)} &= r_t dt - v^f(t,T).dW_t^{Q_\beta} \\ \frac{dE(P_t^i(T)|\tau^i > t)}{E(P_t^i(T)|\tau^i > t)} &= \lambda_t^i dt - v^i(t,T).dW_t^{iQ_T} \end{aligned}$$

The unconditional version of the diffusion equation, shows $P_t^i(T)$ is actually a Q_T martingale as the drift $\lambda_t^i dt$ turns out to compensate the possibility of default. As a martingale, it should be compared to the bond price expressed in the money market account numeraire, which is a martingale, rather than the bond alone:

$$\begin{aligned} B(t,T)/\beta_t &= \exp\left(-\int_0^t r_s ds\right)\exp\left(-\int_t^T f(t,u)du\right) \\ \frac{d\left(B(t,T)/\beta_t\right)}{\left(B(t,T)/\beta_t\right)} &= v^f(t,T).dW_t^{Q_\beta} \end{aligned}$$

A.4.2 The risk neutral measure of choice is Q_T

A second remark concerns the probability measure under which the results hold for credit. While in rates, the risk neutral drift of forward rates only holds in the risk neutral measure associated to this numeraire, the definition (2.4) ensures its martingality under any measure. This being said, it is only under Q_T that these probabilities correspond to the market price of defaultable bonds. While some other derivation of Credit HJM in the literature insist on independence of hazard rates and interest rates for "notational simplicity", this is not needed when working under Q_T.

A.4.3 The filtration of choice is the canonical background filtration

A third remark concerns the filtration under which the equation was derived. Technically, these equations can be derived in any arbitrary subfiltration $\mathcal{F}_t$ of the market filtration $\mathcal{G}_t$. As for the probability measure choice, the General Credit Term Structure Dynamics (2.8) equations remains valid, however, our goal is to obtain the diffusion equations for the market probabilities, not conditional on an arbitrary filtration that warps the hazard rate.

A.5 Some further insight on the (HH1) hypothesis

A.5.1 Implying intensities from hazard rates

We can use the results obtained previously to throw some light at hypothesis (HH1): First, let's recall that intensities can be implied from hazard rates: An application of formula (2.10) is to link the background intensity evolution with the hazard rates dynamics. We have for $\tau^i > t$, $P_t^i(t) = 1$, hence:

$$\int_0^t \lambda_s^i - h^i(0,s)\,ds = \int_0^t \frac{1}{2}\left|v^i(s,t)\right|_\rho^2 ds + v^i(s,t).dW_s^i \quad \text{(A.14)}$$

$$d\lambda_t^i = \frac{\partial h^i}{\partial T}\left(t, T=t^+\right)dt + \sigma^i(t,t).dW_t^i \quad \text{(A.15)}$$

It is also the transposition to credit of HJM formulae for interest rate known to relate the money market account numeraire to the forward diffusion[23]. The equation (A.15) shows the link between the hazard rates, which are an expression of the martingale representation of the survival probability, and the background intensity dynamics.

A.5.2 Implying hazard rates from intensities

To better understand the property to be verified by hazard rates to obtain (HH1), we can choose the measure that makes forwards martingales:

$$\begin{aligned} df(t,T) &= \sigma^f(t,T)dW_t^{Q_T} \\ dh^i(t,T) &= \sigma^i(t,T)dW_t^{\bar{Q}_T^i} \end{aligned}$$

[23] While the HJM equation in the rates case is only true under the risk neutral measure Q_β associated to the money market account numeraire, the credit HJM would hold under any measure, but only relates to market quantities under Q_β.

That is, instead of Q_β for rates and Q_T for credit, we now use Q_T resp. $\bar{Q}^i_T$ (the latter denotes the survival measure for $\tau^i > T$). Going backward from T, we therefore have:

$$\begin{aligned} f(t,T) &= E^{Q_T}(r_T|\mathcal{G}_t) \\ h^i(t,T) &= E^{\bar{Q}^i_T}(\lambda^i_T|\mathcal{G}_t) \end{aligned}$$

This result brings two comments: first, when the literature states the diffusion conditional on survival $\tau > t$, hazard rate drifts by $\sigma^i(t,T).\rho v^i(t,T)dt$, which means that, when compared to our general diffusion formula, although the diffusion has been conditioned to $\tau > t$, but the drift did not change (In fact, it would have changed if it had been conditioned to survival up to any date different from t). Secondly, we notice that the survival measure equation can be rewritten as:

$$h^i(t,T) = E^{\bar{Q}^i_T}(\lambda^i_T|\mathcal{G}_t) = E^{Q_T}(\lambda^i_T|\mathcal{G}_t, \tau^i > T)$$

this can be used for a definition of hazard rates even when hypothesis (HH1) does not hold. As hazard rates diffusion is still essentially determined by $v^i(t,T)$, a sufficient condition for (HH1) now seems to be that the survival probability diffusion term be independent of $\tilde{\theta}^i$.

B Appendix on Diversification Effects

B.1 Diversification in the Independent Systemic Hazard Case

We first have :

$$\begin{aligned} E(L_T|\mathcal{G}_t) &= E(\frac{1}{n}\sum_{i=1}^{n} 1_{\{\tau^i<T\}}|\mathcal{G}_t) = 1 - E(\frac{1}{n}\sum_{i=1}^{n} 1_{\{\tau^i>T\}}|\mathcal{G}_t) \\ &= 1 - \sum_{i=1}^{n}\frac{1}{n}P^i_t(T) = 1 - Q_t(T)\sum_{i=1}^{n}\frac{1}{n}M_i \end{aligned}$$

One can show that risk of $\sum_{i=1}^{n}\frac{1}{n}M_i$ completely diversifies away for a large portfolio. This is not evident since the M_i include terms that are not independent:

$$Var\left(\sum_{i=1}^{n}\frac{1}{n}M_i\right) = \frac{1}{n^2}\sum_{i=1}^{n} Var(M_i) + \frac{1}{n^2}\sum_{i,j} Cov(M_i, M_j) \to 0$$

To see that the covariance is nonetheless 0, we introduce the following notations :

$$M_i = X_iY_iI_i \quad \text{(B.1)}$$

$$X_i := \varepsilon\left(-\int_0^t v^{\epsilon i}(s,T).dW^{\epsilon i}_s\right) \quad \text{(B.2)}$$

$$Y_i := \exp\left(\int_0^t \lambda^i_s ds\right) \quad \text{(B.3)}$$

$$I_i := 1_{\{\tau^i>t\}}$$

$$\mathcal{F}_t := \sigma\left(W^c_s, W^{\epsilon i}_s, \forall i, \forall s < t\right) \quad \text{(B.4)}$$

Assumption (HH2) gives (see A.2) :

$$Cov(I_i, I_j|\mathcal{F}_t) = 0$$

We have :

$$Cov(X_iY_iI_i, X_jY_jI_j|\mathcal{F}_t) = X_iY_iX_jY_jCov(I_i, I_j|\mathcal{F}_t) = 0 \quad \text{(B.5)}$$

$$E(X_iY_iI_i|\mathcal{F}_t) = X_iY_iE(I_i|\mathcal{F}_t) = X_i \quad \text{(B.6)}$$

$$E(M_i) = E(E(M_i|\mathcal{F}_t)) = E(X_i) = 1 \quad \text{(B.7)}$$

then :

$$\begin{aligned} Cov(M_i, M_j) &= Cov(X_iY_iI_i, X_jY_jI_j) && \text{(B.8)} \\ &= E\left(Cov(X_iY_iI_i, X_jY_jI_j|\mathcal{F}_t)\right) \\ &+ Cov(E(X_iY_iI_i|\mathcal{F}_t), E(X_jY_jI_j|\mathcal{F}_t)) && \text{(B.9)} \\ &= E\left(0\right) + Cov(X_i, X_j) && \text{(B.10)} \\ &= 0 && \text{(B.11)} \end{aligned}$$

And:

$$E(L_T|\mathcal{G}_t)\overrightarrow{n\to\infty}1 - Q_t(T)$$

B.2 Diversification in the General Case

To show that the covariance term is null, we need to modify our variables definitions accordingly:

$$\begin{aligned} M_i &:= \frac{P_t^i\left(T\right)}{Q_t\left(T\right)} = X_i \exp\left(\int_0^t \lambda_s^i ds\right) 1_{\{\tau^i > t\}} && \text{(B.12)} \\ X_i &:= \frac{\varepsilon\left(-\int_0^t v_s^i\left(T\right).dW_s^i\right)}{E\left(\varepsilon\left(-\int_0^t v_s^i\left(T\right).dW_s^i\right)\middle|\mathcal{F}_t^c\right)} && \text{(B.13)} \end{aligned}$$

We then have

$$\begin{aligned} Cov(M_i, M_j) &= Cov(X_i, X_j) && \text{(B.14)} \\ &= Cov\left(E\left(X_i|\mathcal{F}_t^c\right), E\left(X_j|\mathcal{F}_t^c\right)\right) + E\left(Cov(X_i, X_j|\mathcal{F}_t^c)\right) && \text{(B.15)} \\ &= 0 && \text{(B.16)} \end{aligned}$$

The first term is 0 because $E\left(X_i|\mathcal{F}_t^c\right) = 1$,the second term is 0 because X_i is $\mathcal{F}_t^i = \sigma\left(W_s^i, s < t\right)$ measurable (resp. $X_j$is $\mathcal{F}_t^j$ measurable), and $\mathcal{F}_t^i, \mathcal{F}_t^j$ are independent conditional on $\mathcal{F}_t^c$,making X_i, X_j independent conditional on $\mathcal{F}_t^c$.

B.3 From Diversification to Independence

The previous section shows that the covariance of the $P_t^i\left(T\right)$ conditional on $Q_t^i(T)$ is 0. However, this is not sufficient for the $P_t^i\left(T\right)$ to be independent. First, we can check what it takes for these variables to be pairwise independent: Independence involves properties of the higher order moments as well, as the variables are bounded in [0,1], existence of all the higher moments is granted. A necessary and sufficient condition for pairwise independence is:

$$\forall r \geq 1, \forall s \geq 1, Cov(P_t^i\left(T\right)^r, P_t^j\left(T\right)^s) = 0 \qquad \text{(B.17)}$$

To explicit what it means for $P_t^i(T)^r$ and $P_t^j(T)^s$ to have 0 covariance conditional on some $\mathcal{F}_t^c$ measurable variable, we can introduce the variables $Q_t^{i,r}(T)$ and $Q_t^{j,s}(T)$:

$$P_t^i(T)^r = P_0^i(T)^r \varepsilon\left(-\int_0^t v_s^i(T).dW_s^i\right)^r \exp\left(\int_0^t \lambda_s^i ds\right)^r 1_{\{\tau^i>t\}} \tag{B.18}$$

$$= P_0^i(T)^r \varepsilon\left(-\int_0^t v_s^i(T).dW_s^i\right)^r \exp\left(\int_0^t \lambda_s^i ds\right)^{r-1} \exp\left(\int_0^t \lambda_s^i ds\right) 1_{\{\tau^i>t\}} \tag{B.19}$$

$$Q_t^{i,r}(T) := E\left(\varepsilon\left(-\int_0^t v_s^i(T).dW_s^i\right)^r \exp\left(\int_0^t \lambda_s^i ds\right)^{r-1} \middle| \mathcal{F}_t^c\right) \tag{B.20}$$

$$M_{i,r} := \frac{P_t^i(T)}{Q_t^{i,r}(T)} = X_{i,r} Y_i I_i \tag{B.21}$$

$$X_{i,r} := \frac{\varepsilon\left(-\int_0^t v_s^i(T).dW_s^i\right)^r \exp\left(\int_0^t \lambda_s^i ds\right)^{r-1}}{E\left(\varepsilon\left(-\int_0^t v_s^i(T).dW_s^i\right)^r \exp\left(\int_0^t \lambda_s^i ds\right)^{r-1} \middle| \mathcal{F}_t^c\right)} \tag{B.22}$$

$$Y_i := \exp\left(\int_0^t \lambda_s^i ds\right) \tag{B.23}$$

$$I_i := 1_{\{\tau^i>t\}} \tag{B.24}$$

We have now:

$$Cov\left(P_t^i(T)^r, P_t^j(T)^s | Q_t^{i,r}(T), Q_t^{j,s}(T)\right) = Q_t^{i,r}(T) Q_t^{j,s}(T) Cov(M_{i,r}, M_{j,s}) \tag{B.25}$$

we can use the previous section demonstration, leading to :

$$Cov\left(P_t^i(T)^r, P_t^j(T)^s | Q_t^{i,r}(T), Q_t^{j,s}(T)\right) = 0 \tag{B.26}$$

We see that the $P_t^i(T)$ and $P_t^j(T)$ have pairwise independent conditional on an infinite set of $\mathcal{F}_t^c$ measurable variables $Q_t^{i,r}(T), Q_t^{j,s}(T), r \geq 1, s \geq 1$.

However, what is actually needed to justify the use of a convolution algorithm is not only pairwise independence but independence of all the tuples of variables. This is an even stronger property, as it involves joint products with all exponents $r_i \geq 0$:

$$E\left(\prod_{i=1}^n P_t^i(T)^{r_i} \middle| Q_t^{i,r_i}(T), i = 1 \dots n\right) = \prod_{i=1}^n E\left(P_t^i(T)^{r_i} \middle| Q_t^{i,r_i}(T), i = 1 \dots n\right) \tag{B.27}$$

A way to do that is to show that for any $P_t^i(T)^{r_i}, P_t^j(T)^{r_j}, P_t^k(T)^{r_k}, \dots$ we have:

$$Cov(P_t^i(T)^{r_i}, P_t^j(T)^{r_j} P_t^k(T)^{r_k} ... | Q_t^{i,r_i}(T), Q_t^{i,r_j}(T), Q_t^{i,r_k}(T), ...) = 0$$

This will prove the independence result by allowing us to factorise the factors one by one (by induction). Our aim is therefore to prove that for any $M_{i,r_i}, M_{j,r_j}, M_{k,r_k}, \dots$ we have:

$$Cov(M_{i,r_i}, M_{j,r_j} M_{k,r_k}, ..) = 0$$

As the properties remain the same we drop the r_i from now. The hypothesis (HH2) gives:

$$Cov(I_i, I_j I_k ... | \mathcal{F}_t) = 0$$

with is a consequence of the tuples of $\tilde{U}^i$ being independent, and it stronger that what was used in Appendix B.1 where we only used their pariwise independence. Thus with the same argument of Appendix B.1, we obtain:

$$Cov(M_i, M_j M_k ...) = Cov(X_i, X_j X_k ...)$$

which becomes as in Appendix B.2:

$$Cov(M_i, M_j M_k...) = Cov\left(E\left(X_i|\mathcal{F}_t^c\right), E\left(X_j X_k...|\mathcal{F}_t^c\right)\right) + E\left(Cov(X_i, X_j X_k...|\mathcal{F}_t^c)\right) \quad (B.28)$$
$$= 0 \quad (B.29)$$

The second term is 0 because X_i is $\mathcal{F}_t^i = \sigma\left(W_s^i, s < t\right)$ measurable (resp. $X_j$is $\mathcal{F}_t^j$ measurable), and $\mathcal{F}_t^i, \mathcal{F}_t^j\hat{}\mathcal{F}_t^j\hat{}...$ are independent conditional on $\mathcal{F}_t^c$,making $X_i, X_j X_k...$ independent conditional on $\mathcal{F}_t^c$. The first term is 0 because $E\left(X_i|\mathcal{F}_t^c\right) = 1$.

We have a much stronger result when $T = t$, as the variable $P_t^i(t) = 1_{\{\tau^i > t\}}$ are bernoulli variables, and can only take the values 0 or 1, and $P_t^i(t)^r = P_t^i(t)$, so that the property:

$$E\left(\prod_{i=1}^n P_t^i(T)^{r_i}\middle| Q_t^i(T), i = 1\dots n\right) = \prod_{i=1}^n E\left(P_t^i(T)^{r_i}\middle| Q_t^i(T), i = 1\dots n\right), \quad i \in \{0,1\} \quad (B.30)$$

is sufficient, for the indicators $1_{\{\tau^i > t\}}$ to be all independent conditional on the $Q_t^i(t)$.

B.4 Existence of a One Factor Copula linked to Degenerate Case

Let's introduce the conditional probability functions:

$$g_1(u) := \Pr(\tau^1 > T|U_1 = u) \quad (B.31)$$
$$g_2(u) := \Pr(\tau^2 > T|U_2 = u) \quad (B.32)$$
$$g_3(u) := \Pr(\tau^1 > T|U = u) \quad (B.33)$$
$$g_4(u) := \Pr(\tau^2 > T|U = u) \quad (B.34)$$

We assume without loss of generality that the variables U, U_1, U_2 have uniform marginals. We have:

$$B_{k,n}(u) := C_n^k u^k (1-u)^{n-k} \quad (B.35)$$
$$P(L_1 = k_1) = \int_u P(L_1 = k_1|U_1 = u)du \quad (B.36)$$
$$= \int_u B_{k_1 n,n}(g_1(u))du \quad (B.37)$$
$$= \int_u B_{k_1 n,n}(u)dF_1(u) \quad (B.38)$$
$$F_1 := g_1^{-1} \quad (B.39)$$
$$P(L_1 = k_1) = \int_u P(L_1 = k_1|U = u)du \quad (B.40)$$
$$= \int_u B_{k_1 n,n}(g_3(u))du \quad (B.41)$$
$$= \int_u B_{k_1 n,n}(u)dF_3(u) \quad (B.42)$$
$$F_3 := g_3^{-1} \quad (B.43)$$

This shows that F_1 and F_3 share the same first n coefficients when they are decomposed on the Bernstein base of polynomials (which spans the set of polynomials). Thus when n tends to infinity, F_1 and F_3 tend to the same large pool density, and we have $g_3^{-1} \to g_1^{-1}$ as $n \to \infty$.

We will also use the following result:

$$\sum_{k > Kn} B_{k,n}(p)\overrightarrow{n \to \infty} 1_{\{p > K\}} \quad (B.44)$$

This can be shown to be a consequence of the convergence in law of the average of iid bernoulli I_i towards a dirac centered on $E(I_i) = p$. The characteristic function of $L^n := \frac{1}{n}\sum_{i=1}^n I_i$ has the following asymptotic behavior:

$$E\left(e^{tL^n}\right) = \prod_{i=1}^{n} E\left(e^{\frac{t}{n}I_i}\right) \quad \text{(B.45)}$$

$$= \left(pe^{\frac{t}{n}} + (1-p)\right)^n \quad \text{(B.46)}$$

$$= \left(1 + \frac{tp}{n} + O\left(\frac{1}{n^2}\right)\right)^n \quad \text{(B.47)}$$

$$\overrightarrow{n\to\infty} \quad e^{tp} = E\left(e^{t\delta_p}\right) \quad \text{(B.48)}$$

The convergence of characteristic functions establishes the convergence in law of L^n towards the dirac distribution L^∞ centered on $E(I_i) = p$. This gives us the convergence of the cumulative distribution functions:

$$L^n \quad \overrightarrow{law} \quad L^\infty \quad \text{(B.49)}$$

$$P\left(L^n > K\right) \quad \overrightarrow{n\to\infty} \quad P\left(L^\infty > K\right) = \int_k^1 \delta_{\{x=p\}}dx = 1_{\{p>K\}} \quad \text{(B.50)}$$

$$\sum_{k>Kn} B_{k,n}(p) \quad \overrightarrow{n\to\infty} \quad 1_{\{p>K\}} \quad \text{(B.51)}$$

The joint distribution of L_1 and L_2 can be explicited using the independence of losses conditional on (U_1, U_2) :

$$P(L_1 > k_1, L_2 > k_2) = \int_{u_1,u_2} P(L_1 > k_1|U_1 = u_1)P(L_2 > k_2|U_2 = u_2)f(u_1,u_2)du_1du_2 \quad \text{(B.52)}$$

$$= \int_{u_1,u_2} \sum_{i>k_1n} B_{i,n}(g_1(u_1)) \sum_{j>k_2n} B_{j,n}(g_2(u_2))f(u_1,u_2)du_1du_2 \quad \text{(B.53)}$$

$$P(L_1 > k_1, L_2 > k_2)\overrightarrow{n\to\infty} \int_{u_1,u_2} 1_{\{g_1(u_1)>k_1\}}1_{\{g_2(u_2)>k_2\}}f(u_1,u_2)du_1du_2$$

Therefore, the joint density converges towards f_{L_1,L_2}, given by:

$$\frac{\partial^2 P(L_1 > k_1, L_2 > k_2)}{\partial k_1 \partial k_2} \quad \overrightarrow{n\to\infty} \quad f_{L_1,L_2}(k_1,k_2) \quad \text{(B.54)}$$

$$f_{L_1,L_2}(k_1,k_2) = g_1^{-1\prime}(k_1)g_2^{-1\prime}(k_2)f(g_1^{-1}(k_1), g_2^{-1}(k_2)) \quad \text{(B.55)}$$

$$= F_1'(k_1)F_2'(k_2)f(F_1(k_1), F_2(k_2)) \quad \text{(B.56)}$$

If we do the same using conditional independence on the single factor variable U, we have:

$$P(L_1 > k_1, L_2 > k_2) = \int_u P(L_1 = k_1|U = u)P(L_2 = k_2|U = u)f(u)du \quad \text{(B.57)}$$

$$= \int_u \sum_{i>k_1n} B_{i,n}(g_3(u)) \sum_{j>k_2n} B_{j,n}(g_4(u))f(u)du \quad \text{(B.58)}$$

$$\overrightarrow{n\to\infty} \quad \int_u 1_{\{g_3(u)>k_1\}}1_{\{g_4(u)>k_2\}}f(u)du \quad \text{(B.59)}$$

This means that the joint distribution is concentrated on the domain defined by:

$$C = \left\{g_3^{-1}(k_1) = g_4^{-1}(k_2) = u, u \in [0,1]\right\}$$

and the density on this curve is the factor density f:

$$\frac{\partial P(L_1 > g_3(u), L_2 > g_4(u))}{\partial u}\overrightarrow{n\to\infty}f(u) \quad \text{(B.60)}$$

The joint loss density converges to 0 outside this curve. The asymptotic joint distribution is:

$$f_{L_1,L_2}(k_1,k_2) = g_3^{-1\prime}(k_1) g_4^{-1\prime}(k_2) \delta_{\left\{g_3^{-1}(k_1)=g_4^{-1}(k_2)=u\right\}} f(u) \tag{B.61}$$

Thus, for any point (k_1, k_2) where the limit joint density of loss is not 0, there must exists u such that, $g_3^{-1}(k_1) \to u, g_4^{-1}(k_2) \to u$ as $n \to \infty$. Which means that $g_3^{-1}(k_1) - g_4^{-1}(k_2) \to 0$ as $n \to \infty$ is a necessary condition to avoid a 0 density at this point.

As the two factors are not comonotonic, the distribution of L_1 conditional on L_2 is not concentrated in a dirac, and therefore, there exists k_2, k_1, k_1' such that:

$$k_1' \neq k_1, f_{L_1,L_2}(k_1,k_2) > 0, f_{L_1,L_2}\left(k_1', k_2\right) > 0$$

Hence:

$$g_3^{-1}(k_1) - g_4^{-1}(k_2) \quad \overrightarrow{n \to \infty} \quad 0 \tag{B.62}$$
$$g_3^{-1}(k_1') - g_4^{-1}(k_2) \quad \overrightarrow{n \to \infty} \quad 0 \tag{B.63}$$
$$g_3^{-1}(k_1) - g_3^{-1}(k_1') \quad \overrightarrow{n \to \infty} \quad 0 \tag{B.64}$$
$$g_1^{-1}(k_1) - g_1^{-1}(k_1') \quad \overrightarrow{n \to \infty} \quad 0 \tag{B.65}$$

which is not possible for a smooth increasing function g_1.

B.5 Convergence towards the canonical copula

The intuition of the proof is that the first equations B.37 and B.42 of the previous demonstration shows that if a group of n identical names has two different admissible copulas, these two conditional probability functions associated to these two copulas share the same first n coefficients when they are decomposed on the Bernstein base of polynomials (which spans the set of polynomials). We now give more details.

On one group of n identical names, let's assume we can find two different copulas, and use the previous notations :

$$g_1(u) := \quad \Pr(\tau^1 > T | U_1 = u) \tag{B.66}$$
$$g_3(u) := \quad \Pr(\tau^1 > T | U = u) \tag{B.67}$$
$$F_1 \ : \ = g_1^{-1} \tag{B.68}$$
$$F_3 \ : \ = g_3^{-1} \tag{B.69}$$

where U and $U_1$are two uniforms variables, g_3 and g_1 are the two conditional probability functions, that map the copulas and the uniform variables. F_1 and F_3 thus are the density functions of the two copulas.

As we already know that the canonical copula is always an admissible copula, let's assume that F_1 is the canonical copula distribution and F_3 another copula.

Then :

$$B_{k,n}(u) := C_n^k u^k (1-u)^{n-k} \tag{B.70}$$

$$P(L_1 = k_1) = \int_u P(L_1 = k_1 | U_1 = u) du \tag{B.71}$$

$$= \int_u B_{k_1 n,n}(g_1(u)) du \tag{B.72}$$

$$= \int_u B_{k_1 n,n}(u) dF_1(u) \tag{B.73}$$

$$P(L_1 = k_1) = \int_u P(L_1 = k_1 | U = u) du \tag{B.74}$$

$$= \int_u B_{k_1 n,n}(g_3(u)) du \tag{B.75}$$

$$= \int_u B_{k_1 n,n}(u) dF_3(u) \tag{B.76}$$

The same considerations show that F_1 and F_3 share the same first n Bernstein coefficients.

Thus when n tends to infinity, F_1 and F_3 tend to the same large pool density, therefore the two copulas become identical when n tends to infinity, and all copulas tend to the canonical copula.